On the foundations of nonlinear generalized functions II

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Abstract. This paper gives a comprehensive analysis of algebras of Colombeau-type generalized functions in the range between the diffeomorphism-invariant quotient algebra $\mathcal{G}^d = \mathcal{E}_M/\mathcal{N}$ introduced in part I and Colombeau's original algebra $\mathcal{G}^e$. Three main results are established: First, a simple criterion describing membership in $\mathcal{N}$ (applicable to all types of Colombeau algebras) is given. Second, two counterexamples demonstrate that $\mathcal{G}^d$ is not injectively included in $\mathcal{G}^e$. Finally, it is shown that in the range "between" $\mathcal{G}^d$ and $\mathcal{G}^e$ only one more construction leads to a diffeomorphism invariant algebra. In analyzing the latter, several classification results essential for obtaining an intrinsic description of $\mathcal{G}^d$ on manifolds are derived.

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12 Introduction to part II

In the present article which is the second in a series of two, we continue the study of diffeomorphism invariant Colombeau algebras. We will use freely notation and results from the first part ([21]); the latter will be referred to herein simply as "Part I". Also, numbering of sections, theorems and formulas will be continued.

The main result of section 13 permits one to simplify the definition of the ideal $\mathcal{N}$ considerably: It dispenses with taking into account the derivatives of the representative being tested. This applies to virtually all versions of Colombeau algebras. This seemingly technical modification, however, has decisive effects on applications: For example, it makes it considerably easier to prove uniqueness of the solutions of many differential equations. Section 14 complements section 4 ("Calculus") of Part I by certain results needed in section 15. In particular, it is shown that $\mathcal{C}^\infty(U, F)$ is complete with respect to the topology of uniform convergence (on a suitable family of bounded sets) in all derivatives resp. differentials, provided F is complete as a locally convex space. In section 15 we show that the diffeomorphism invariant algebra $\mathcal{G}^d(\Omega)$ presented in section 7 of Part I is not injectively included in the Colombeau algebra $\mathcal{G}^e(\Omega)$ of [10] (which, to be sure, is the standard version among those being

independent from the choice of a particular approximation of the delta distribution) by constructing two counterexamples. In section 16 we develop a framework allowing to classify the range of algebras which can be positioned between $\mathcal{G}^d(\Omega)$ and (the smooth version of) $\mathcal{G}^e(\Omega)$. In particular, we are going to discuss to which extent at least the definition of the algebra introduced by J. F. Colombeau and A. Meril in [13] has to be modified to obtain diffeomorphism invariance. This leads to the construction of the (diffeomorphism invariant) Colombeau algebra $\mathcal{G}^2(\Omega)$ which is closer to the algebra of [13] than the algebra $\mathcal{G}^d(\Omega)$ (section 17). Certain classification results of sections 16 and 17 are essential for obtaining an intrinsic description of Colombeau algebras on manifolds (see [26]). The concluding section 18 points out that also weaker invariance properties than with respect to all diffeomorphisms should be envisaged for Colombeau algebras, in particular regarding applications.

In the following, we will abbreviate $R \circ S^{(\varepsilon)}$ as R_ε, throughout. Terms of the form $\partial^\alpha \mathrm{d}_1^k R_\varepsilon$ always are to be read as $\partial^\alpha \mathrm{d}_1^k(R_\varepsilon)$.

13 A simple condition equivalent to negligibility

The property of a representative $R \in \mathcal{E}(\Omega)$ of a generalized function $[R] \in \mathcal{G}(\Omega)$ to belong to the ideal $\mathcal{N}(\Omega)$ was defined in 7.3. Theorem 18 (2°) of [28] resp. Theorem 7.13 of section 7 give an equivalent condition replacing the term $\partial^\alpha(R(S_\varepsilon\phi(\varepsilon,x),x))$ occurring in 7.3 by $(\partial^\alpha \mathrm{d}_1^k R_\varepsilon)(\varphi,x)(\psi_1,\dots,\psi_k)$. Moreover, Theorem 18 (1°) of [28] shows that we still get a condition equivalent to $R \in \mathcal{N}(\Omega)$ if we simply omit the differential with respect to the first variable φ from the statement of (2°), provided R is assumed to be moderate. In the following, we are going to show that a further simplification is possible which might seem rather drastic at first glance: It is not even necessary to consider partial derivatives with respect to $x \in \Omega$. In order to facilitate comparing the conditions mentioned so far we include all of them in the following theorem, though only (0°) is new.

13.1 Theorem. *Let Ω be an open subset of $\mathbb{R}^s$ and $R \in \mathcal{E}_M(\Omega)$. Then each of the following conditions is equivalent to $R \in \mathcal{N}(\Omega)$ (in the sense of 7.3):*

(0°) $\forall K \subset\subset \Omega\ \forall n \in \mathbb{N}\ \exists q \in \mathbb{N}\ \forall B$ *(bounded)* $\subseteq \mathcal{D}(\mathbb{R}^s)$*:*

$$R_\varepsilon(\varphi,x) = O(\varepsilon^n) \qquad (\varepsilon \to 0)$$

uniformly for $x \in K$, $\varphi \in B \cap \mathcal{A}_q(\mathbb{R}^s)$.

(1°) $\forall K \subset\subset \Omega\ \forall \alpha \in \mathbb{N}_0^d\ \forall n \in \mathbb{N}\ \exists q \in \mathbb{N}\ \forall B$ *(bounded)* $\subseteq \mathcal{D}(\mathbb{R}^s)$*:*

$$\partial^\alpha R_\varepsilon(\varphi,x) = O(\varepsilon^n) \qquad (\varepsilon \to 0)$$

uniformly for $x \in K$, $\varphi \in B \cap \mathcal{A}_q(\mathbb{R}^s)$.

(2°) $\forall K \subset\subset \Omega\ \forall \alpha \in \mathbb{N}_0^d\ \forall k \in \mathbb{N}_0\ \forall n \in \mathbb{N}\ \exists q \in \mathbb{N}\ \forall B\,(bounded) \subseteq \mathcal{D}(\mathbb{R}^s)$:

$$\partial^\alpha \mathrm{d}_1^k R_\varepsilon(\varphi, x)(\psi_1, \dots, \psi_k) = O(\varepsilon^n) \qquad (\varepsilon \to 0)$$

uniformly for $x \in K$, $\varphi \in B \cap \mathcal{A}_q(\mathbb{R}^s)$, $\psi_1, \dots, \psi_k \in B \cap \mathcal{A}_{q0}(\mathbb{R}^s)$.

Proof. The equivalence of each of (1°) and (2°) with $R \in \mathcal{N}(\Omega)$ is a part of Theorem 18 of [28]. $(1^\circ) \Rightarrow (0^\circ)$ being trivial, it remains to show $(0^\circ) \Rightarrow (1^\circ)$. To this end, we will prove, assuming $R \in \mathcal{E}_M(\Omega)$ to satisfy (0°), that R satisfies (1°) for $\alpha := e_i$, i.e., $\partial^\alpha = \partial_i$ $(i = 1, \dots, s)$ and that, in addition, $\partial_i R$ again is moderate and satisfies (0°). Then it will follow by induction that (1°) holds for all $\alpha \in \mathbb{N}_0^s$.

So suppose $R \in \mathcal{E}_M(\Omega)$ to satisfy (0°) and let $K \subset\subset \Omega$ and $n \in \mathbb{N}$ be given. For $\delta := \min(1, \mathrm{dist}(K, \partial\Omega))$, set $L := K + \overline{B}_{\frac{\delta}{2}}(0)$. Then $K \subset\subset L \subset\subset \Omega$. Now by moderateness of R and Theorem 7.12, choose $N \in \mathbb{N}$ such that for every bounded subset B of $\mathcal{D}(\mathbb{R}^s)$ the relation $\partial_i^2 R_\varepsilon(\varphi, x) = O(\varepsilon^{-N})$ as $\varepsilon \to 0$ holds, uniformly for $x \in L$, $\varphi \in B \cap \mathcal{A}_0(\mathbb{R}^s)$. Next, by the assumption of (0°) to hold for R, choose $q \in \mathbb{N}$ such that, again for every bounded subset B of $\mathcal{D}(\mathbb{R}^s)$, we have $R_\varepsilon(\varphi, x) = O(\varepsilon^{2n+N})$ as $\varepsilon \to 0$, uniformly for $x \in L$, $\varphi \in B \cap \mathcal{A}_q(\mathbb{R}^s)$. Now suppose a bounded subset B of $\mathcal{D}(\mathbb{R}^s)$ to be given; let $\varphi \in B \cap \mathcal{A}_q(\mathbb{R}^s)$, $x \in K$ and $0 < \varepsilon < \frac{\delta}{2}$; hence $x + \varepsilon^{n+N} e_i \in L$. By Taylor's theorem, we conclude (to be precise, separately for the real and imaginary part of R)

$$R_\varepsilon(\varphi, x + \varepsilon^{n+N} e_i) \;=\; R_\varepsilon(\varphi, x) + \partial_i R_\varepsilon(\varphi, x)\varepsilon^{n+N} + \frac{1}{2}\partial_i^2 R_\varepsilon(\varphi, x_\theta)\varepsilon^{2n+2N}$$

where $x_\theta = x + \theta\varepsilon^{n+N} e_i$ for some $\theta \in (0,1)$; note that also $x_\theta \in L$. Consequently,

$$\partial_i R_\varepsilon(\varphi, x) \;=\; \underbrace{\big(R_\varepsilon(\varphi, x + \varepsilon^{n+N} e_i) - R_\varepsilon(\varphi, x)\big)}_{O(\varepsilon^{2n+N})}\varepsilon^{-n-N} - \underbrace{\frac{1}{2}\partial_i^2 R_\varepsilon(\varphi, x_\theta)}_{O(\varepsilon^{-N})}\varepsilon^{n+N},$$

uniformly for $\varphi \in B \cap \mathcal{A}_q(\mathbb{R}^s)$, $x \in K$. Having demonstrated $\partial_i R_\varepsilon(\varphi, x) = O(\varepsilon^n)$ for all $i = 1, \dots, s$, observe that $\partial_i(R_\varepsilon) = (\partial_i R)_\varepsilon$. Therefore, $\partial_i R$ again satisfies (0°). According to Theorem 7.10 (which is non-trivial, see the discussion in section 7), $\partial_i R$ is also moderate . By the remark made above, this completes the proof. □

The reader acquainted with E. Landau's paper [33] will easily recognize the method employed therein to form the basis of the preceding proof (though not mentioned explicitly in [28], this equally applies to the proof of $(1^\circ) \Rightarrow (2^\circ)$ of Theorem 18 of [28]).

The part of Theorem 13.1 saying that for moderate functions (the appropriate analog of) condition (0°) is equivalent to negligibility applies to virtually all versions of Colombeau algebras of practical importance, in particular, to the following:

- For the special algebra as defined, e.g., in [35], p. 109, just replace the term $R_\varepsilon(\varphi, x)$ in condition (0°) by $u_\varepsilon(x)$.
- For the classical full Colombeau algebra of [10] simply drop the uniformity requirement concerning φ from (0°).
- For the diffeomorphism invariant Colombeau algebra $\mathcal{G}^2(\Omega)$ to be introduced in section 17, the corresponding result is stated as Theorem 17.9.
- For the special algebra on smooth mainfolds the corresponding result follows from the local characterization of generalized functions (see [41], 4.2).
- The latter also applies to the intrinsically defined full Colombeau algebra on manifolds ([26], Corollary 4.5).

In the first and second of these four instances, the respective proofs are obtained by appropriately slimming down the corresponding argument of the proof of Theorem 13.1.

The seemingly technical difference between (0°) and the remaining conditions (including negligibility of R) has decisive effects on applications: For example, if the uniqueness of a solution of a differential equation is to be shown one supposes R_1, R_2 to be representatives of solutions. Note that this includes the assumption that $R_1, R_2 \in \mathcal{E}_M(\Omega)$, hence Theorem 13.1 may be applied. For $[R_1] = [R_2]$ in $\mathcal{G}(\Omega)$ we have to show that $R := R_1 - R_2 \in \mathcal{N}(\Omega)$. Now it suffices to check condition (0°) rather than (1°) (resp. (2°) resp. the original definition of $R \in \mathcal{N}(\Omega)$), i.e., there is no need to analyze the behaviour of any derivative of R.

Apart from that, condition (0°) is also of theoretical relevance. To give a sample, let us demonstrate that it allows to simplify considerably the proof of statement (iv) of Theorem 7.4 in Part I (saying that $(\iota - \sigma)(\mathcal{C}^\infty(\Omega)) \subseteq \mathcal{N}(\Omega)$): Since $\iota f - \sigma f \in \mathcal{E}_M(\Omega)$ by (i) and (ii) of Theorem 7.4, it is sufficient for $\iota f - \sigma f \in \mathcal{N}(\Omega)$ to show that

$$(\iota f - \sigma f)(S_\varepsilon\varphi, x) = \int\limits_{\frac{\Omega - x}{\varepsilon}} [f(z\varepsilon + x) - f(x)]\,\varphi(z)\,dz = O(\varepsilon^{q+1}),$$

uniformly for $x \in K$ and φ ranging over some bounded subset of $\mathcal{A}_q(\mathbb{R}^d)$. This, however, is immediate.

14 Some more calculus

Both the counterexamples to be constructed in section 15 will take the form of infinite series, being absolutely convergent in each derivative. Thus we need a theorem

guaranteeing the completeness of $\mathcal{E}(\Omega) = \mathcal{C}^\infty(U(\Omega), \mathbb{C}) \equiv \mathcal{C}^\infty(\mathcal{A}_0(\Omega) \times \Omega, \mathbb{C})$ with respect to the corresponding topology. The remarkable ease of the proof of this generalization of a standard result of elementary real analysis clearly exhibits the virtues of calculus in convenient vector spaces as outlined in section 4. To this end, let E, F be locally convex spaces and U an open subset of E. If $f : U \to F$ is smooth, its n-th differential $\mathrm{d}^n f$ belongs to $\mathcal{C}^\infty(U, L^n(E^n, F))$ where $L^n(E^n, F)$ denotes the space $L(E, \ldots, E; F)$ of n-linear bounded maps from $E \times \cdots \times E$ (n factors) into F. (For $n = 0$, set $L^n(E^n, F) := F$.) On $\mathcal{C}^\infty(U, L^n(E^n, F))$, let τ^n_{cb} denote the topology of uniform F-convergence on subsets of the form $K \times B$ where K is a compact subset of U and B is bounded in $E^n = E \times \cdots \times E$. Let $\mathcal{C}^\infty(U, F)$ carry the initial (locally convex) topology τ^∞ induced by the family $(\mathrm{d}^n, \mathcal{C}^\infty(U, L^n(E^n, F)), \tau^n_{cb})_{n \geq 0}$, i.e., the topology of uniform convergence of all derivatives (that is to say, differentials) on sets $K \times B$ as above. Note that on $\mathcal{C}^\infty(\mathbb{R}, F)$, τ^∞ is just the usual Fréchet topology of compact convergence in all derivatives.

14.1 Theorem. *Let E, F be locally convex spaces, assume F to be complete and let U be an open subset of E. Then $\mathcal{C}^\infty(U, F)$ is complete with respect to the topology τ^∞ of uniform F-convergence of all differentials on subsets of the form $K \times B$ where K is a compact subset of U and B is bounded in the appropriate product $E^n = E \times \cdots \times E$. Moreover, for each $p \in \mathbb{N}$, the operator $\mathrm{d}^p : \mathcal{C}^\infty(U, F) \to \mathcal{C}^\infty(U, L^p(E^p, F))$ is continuous if both the domain and the range space carry the respective topology τ^∞.*

Proof. Let (f_ι) be a net in $\mathcal{C}^\infty(U, F)$ which is Cauchy with respect to τ^∞, that is, suppose $(\mathrm{d}^n f_\iota)$ to be a Cauchy net in $\mathcal{C}^\infty(U, L^n(E^n, F))$ with respect to τ^n_{cb} for each $n = 0, 1, 2, \ldots$. Due to the completeness of F, each net $(\mathrm{d}^n f_\iota)$ has a limit $f^{[n]} : U \times E^n \to F$ with respect to (the obvious extension of) τ^n_{cb}. In particular, (f_ι) converges to some function $f := f^{[0]} : U \to F$. Consider a smooth curve $c : \mathbb{R} \to U$; then for each ι, $f_\iota \circ c$ is smooth from $\mathbb{R}$ to F, its n-th derivative at $t \in \mathbb{R}$ being given as a certain sum of terms of the form $(\mathrm{d}^l f_\iota)(c(t))(c^{(k_1)}(t), \ldots, c^{(k_l)}(t))$ where $1 \leq l \leq n$ and $\sum k_j = n$, due to the chain rule. With t ranging over some compact subset of $\mathbb{R}$, the values attained by $c^{(k)}(t)$ form a compact subset of U resp. E, for each $k \in \mathbb{N}_0$. Now it follows from the Cauchy property of (f_ι) that $(f_\iota \circ c)$ is Cauchy in $\mathcal{C}^\infty(\mathbb{R}, F)$ with respect to uniform convergence in all derivatives on compact sets. From the completeness of the latter space we conclude that the limit of $(f_\iota \circ c)$ exists as a smooth function and is equal to $f \circ c$. This argument being valid for any smooth curve c, f itself is smooth. To establish $f = \lim f_\iota$ with respect to τ^∞, it remains to show that for any $n \in \mathbb{N}$, $\mathrm{d}^n f = f^{[n]}$, i.e., that for all $x \in U$, $v_1, \ldots, v_n \in E$ we have

$$(\mathrm{d}^n f)(x)(v_1, \ldots, v_n) = \lim\, (\mathrm{d}^n f_\iota)(x)(v_1, \ldots, v_n), \qquad (1)$$

For a straight line $c(t) = x + tv$ we obtain, at $t = 0$, $(g \circ c)^{(n)}(0) = (\mathrm{d}^n g)(x)(v, \ldots, v)$ for any $g \in \mathcal{C}^\infty(U, F)$. Therefore,

$$(\mathrm{d}^n f)(x)(v, \ldots, v) = (f \circ c)^{(n)}(0) = \lim (f_\iota \circ c)^{(n)}(0) = \lim (\mathrm{d}^n f_\iota)(x)(v, \ldots, v).$$

Equation (1) now follows by polarization (see, e.g., [30], Lemma 7.13 (1)). Finally, the continuity of d^p with respect to the initial topologies τ^∞ is immediate from the following commutative diagram:

$$\begin{array}{ccc} (\mathcal{C}^\infty(U, F), \tau^\infty) & \xrightarrow{\mathrm{d}^p} & (\mathcal{C}^\infty(U, L^p(E^p, F)), \tau^\infty) \\ \Big\downarrow \mathrm{d}^{p+n} & & \Big\downarrow \mathrm{d}^n \\ (\mathcal{C}^\infty(U, L^{p+n}(E^{p+n}, F)), \tau_{cb}^{p+n}) & \xrightarrow{\mathrm{id}} & (\mathcal{C}^\infty(U, L^n(E^n, L^p(E^p, F))), \tau_{cb}^n) \end{array}$$

Observe that the lower horizontal arrow is a linear homeomorphism, due to $L^p(E^p, F)$ carrying the topology of uniform convergence on bounded sets. □

For the rest of this section, let U denote a (non-empty) open subset of a closed affine subspace E_1 of some locally convex space E, E_0 the linear subspace parallel to E_1 and F a complete locally convex space. *Mutatis mutandis*, Theorem 14.1 is valid also in this slightly more general situation. The vectors $v_1, \ldots, v_n$ to be plugged into $\mathrm{d}^n f(x)$ now have to be taken from E_0, as well as B has to denote a bounded subset of E_0^n.

14.2 Definition. *Assume, in addition to the above, that the topology of F is generated by some family $\mathcal{P}$ of semi-norms. For fixed $n \in \mathbb{N}_0$, let $(f_k)_{k \in \mathbb{N}}$ denote a sequence of functions*

$$\begin{array}{rcll} f_k : U \times E_0 \times \cdots \times E_0 & \to & F & \qquad (n \text{ factors } E_0) \\ f_k : (\ x\ ,\ v_1\ , \ldots,\ v_n\ ) & \mapsto & f_k(x)(v_1, \ldots, v_n). & \end{array}$$

We say that (f_k) is exponentially bounded *on $U \times E_0^n$ (an* (eb)*-sequence, for short) if for each compact subset K of U, each bounded subset B of E_0 and each $p \in \mathcal{P}$ there exists a constant $C (\geq 1)$ such that $p(f_k(x)(v_1, \ldots, v_n)) \leq C^k$ for any $k \in \mathbb{N}$, $x \in K$ and $v_i \in B$ $(i = 1, \ldots, n)$.*

Define $(f_k) + (g_k) := (f_k + g_k)$ and $\lambda(f_k) := (\lambda f_k)$ $(\lambda \in \mathbb{C})$, as well as $(f_k) \cdot (g_k) := (f_k \cdot g_k)$ provided F is a (complete) locally convex topological algebra. Then the following proposition is immediate, due to $C_1^k + C_2^k \leq (C_1 + C_2)^k$ and $C_1^k C_2^k = (C_1 C_2)^k$:

14.3 Proposition. *The set of* (eb)*-sequences forms a linear space (resp. an algebra if F is a locally convex algebra with jointly sequentially continuous multiplication) with respect to the operations defined above.*

14.4 Theorem. *Let $f_k \in \mathcal{C}^\infty(U,F)$ for every $k \in \mathbb{N}$. Assume that for each fixed $n \in \mathbb{N}_0$, $(\mathrm{d}^n f_k)_k$ is (eb) on $U \times E_0^n$. Then $\sum\limits_{k=0}^{\infty} \frac{1}{k!} f_k$ is convergent with respect to τ^∞ to some $f \in \mathcal{C}^\infty(U,F)$. Moreover, $\mathrm{d}^n f = \sum\limits_{k=0}^{\infty} \frac{1}{k!} \mathrm{d}^n f_k$ for every $n \in \mathbb{N}_0$ where also the latter series converges with respect to τ^∞.*

Proof. Fix $n \in \mathbb{N}_0$, a compact subset K of U and a bounded subset B of E_0. Since $(\mathrm{d}^n f_k)_k$ is (eb), $\sum\limits_k \frac{1}{k!} \mathrm{d}^n f_k$ is majorized, uniformly on $K \times B^n$, by $\sum\limits_k \frac{C_n^k}{k!}$ for some constant $C_n (\geq 1)$ depending only on n, K and B. Consequently, $\sum\limits_k \frac{1}{k!} f_k$ is Cauchy with respect to τ^∞. Now both the convergence of $\sum\limits_k \frac{1}{k!} f_k$ and the admissibility of term-wise differentiation follow from Theorem 14.1. $\square$

In the sequel, 14.2–14.4 will only be used for $F = \mathbb{C}$; the extension to locally convex algebras being for free virtually, we chose to state them in the general form to indicate the scope of Theorem 14.4.

15 Non-injectivity of the canonical homomorphism from $\mathcal{G}^d(\Omega)$ into $\mathcal{G}^e(\Omega)$

For every open subset Ω of $\mathbb{R}^s$, there is a canonical algebra homomorphism Φ from the diffeomorphism invariant Colombeau algebra $\mathcal{G}^d(\Omega)$ of [28] (see section 7) to the "classical" (full) Colombeau algebra $\mathcal{G}^e(\Omega)$ introduced in [10], 1.2.2 (the upper subscript e being taken from the title "Elementary Introduction to New Generalized Functions" of the latter monograph). In this section, we are going to show that Φ is not injective in general by constructing a representative R of a generalized function $[R] \in \mathcal{G}^d(\Omega)$ such that $[R] \neq 0$, yet $\Phi[R] = 0$.

By superscripts d, e we will distinguish between ingredients (as listed in section 3) for constructing $\mathcal{G}^d(\Omega)$ resp. $\mathcal{G}^e(\Omega)$. Observe that superscripts d, e are independent of superscripts J, C as introduced in section 5: Each of the (non-isomorphic) algebras $\mathcal{G}^d(\Omega)$, $\mathcal{G}^e(\Omega)$ has equivalent descriptions in the C- and the J-formalism, respectively. As in section 7 of Part I, we will use the C-formalism also in the present context. All the relevant definitions are to be found in section 7 (for $\mathcal{G}^d(\Omega)$) resp. [10] (for $\mathcal{G}^e(\Omega)$). For the present purpose, the following of them are of particular importance:

$$\begin{aligned} U^d(\Omega) &:= T^{-1}(\mathcal{A}_0(\Omega) \times \Omega) \\ U^e(\Omega) &:= T^{-1}(\mathcal{A}_1(\Omega) \times \Omega)^1 \end{aligned} \qquad (2)$$

$$\begin{aligned}\mathcal{E}^d(\Omega) &:= \mathcal{C}^\infty(U^d(\Omega))\\ \mathcal{E}^e(\Omega) &:= \{R : U^e(\Omega) \to \mathbb{C} \mid x \mapsto R(\varphi, x) \text{ is smooth on } U_\varphi \text{ for each } \varphi\}\end{aligned}$$

where U_φ denotes the (open) set $\{x \mid (\varphi, x) \in U^e(\Omega)\}$.

From now on, we will omit specifying Ω explicitly whenever it is clear which domain is intended. Let $j : U^e \to U^d$ denote set-theoretic inclusion. To see that the restriction map $\Phi_0 = j^*$ maps $\mathcal{E}^d$ into $\mathcal{E}^e$ we have to pass from C-representatives to J-representatives: Smoothness of $R^d \in \mathcal{E}^d$, by definition, is equivalent to smoothness of $(T^*)^{-1}R^d \in \mathcal{C}^\infty(\mathcal{A}_0(\Omega) \times \Omega)$ while for $R^e \in \mathcal{E}^e$, smoothness of $x \mapsto R^e(\varphi, x)$ is equivalent to smoothness of $x \mapsto (T^*)^{-1}R^e(\varphi(.-x), x)$. From this it is clear that $\Phi_0 R^d \in \mathcal{E}^e$ for $R^d \in \mathcal{E}^d$.

Φ_0 even maps $\mathcal{E}^d_M$ into $\mathcal{E}^e_M$ and $\mathcal{N}^d$ into $\mathcal{N}^e$, respectively. This follows easily by inspecting the corresponding definitions: For $R^d \in \mathcal{E}^d$, $R^e \in \mathcal{E}^e$; we have, by definition (omitting the quantifiers "$\forall K \subset\subset \Omega\ \forall \alpha \in \mathbb{N}_0^s\ \exists N \in \mathbb{N}$"),

$$\begin{aligned} R^d \in \mathcal{E}^d_M &\Leftrightarrow \forall \phi \in \mathcal{C}^\infty_b(I \times \Omega, \mathcal{A}_0(\mathbb{R}^s)) : \sup_{x \in K} |\partial^\alpha(R^d(S_\varepsilon \phi(\varepsilon, x), x))| = O(\varepsilon^{-N})\\ R^e \in \mathcal{E}^e_M &\Leftrightarrow \forall \varphi \in \mathcal{A}_N(\mathbb{R}^s) : \sup_{x \in K} |\partial^\alpha(R^e(S_\varepsilon \varphi, x))| = O(\varepsilon^{-N})\end{aligned}$$

Obviously, each test object $\varphi \in \mathcal{A}_N(\mathbb{R}^s)$ can be viewed as a particular case of a test object $\phi \in \mathcal{C}^\infty_b(I \times \Omega, \mathcal{A}_0(\mathbb{R}^s))$ by setting $\phi(\varepsilon, x) := \varphi$ independently of ε, x. Thus from $R^d \in \mathcal{E}^d_M$ it follows that $\Phi_0 R^d \in \mathcal{E}^e_M$. A similar argument shows that $\Phi_0 R^d \in \mathcal{N}^e$ provided $R^d \in \mathcal{N}^d$. Note that the condition for the membership of R^e in $\mathcal{N}^e$ as given in [10], 1.1.11, that is (this time omitting "$\forall K \subset\subset \Omega\ \forall \alpha \in \mathbb{N}_0^s$")

$$\exists N\ \exists \gamma : \mathbb{N} \to \mathbb{R}\ \forall q \geq N\ \forall \varphi \in \mathcal{A}_q(\mathbb{R}^s) : \sup_{x \in K} |\partial^\alpha(R^e(S_\varepsilon \varphi, x))| = O(\varepsilon^{\gamma(q) - N})$$

(where $\gamma(q) \nearrow \infty$) is easily seen to be equivalent to

$$\forall n\ \exists q \qquad \forall \varphi \in \mathcal{A}_q(\mathbb{R}^s) : \sup_{x \in K} |\partial^\alpha(R^e(S_\varepsilon \varphi, x))| = O(\varepsilon^n)$$

which has the same structure as the condition in 7.3 for R^d to belong to $\mathcal{N}^d$:

$$\forall n\ \exists q \qquad \forall \phi \in \mathcal{C}^\infty_b(I \times \Omega, \mathcal{A}_q(\mathbb{R}^s)) : \sup_{x \in K} |\partial^\alpha(R^d(S_\varepsilon \phi(\varepsilon, x), x))| = O(\varepsilon^n)\ .$$

Due to the invariance of $\mathcal{E}_M$ and $\mathcal{N}$ under Φ_0, Φ_0 induces a map $\Phi : \mathcal{G}^d(\Omega) \to \mathcal{G}^e(\Omega)$ acting on representatives as restriction from $T^{-1}(\mathcal{A}_0(\Omega) \times \Omega)$ to $T^{-1}(\mathcal{A}_1(\Omega) \times \Omega)$. Φ is an algebra homomorphism respecting the embeddings of $\mathcal{D}'(\Omega)$ and differentiation.

[1]The choice of $\mathcal{A}_1(\Omega)$ rather than $\mathcal{A}_0(\Omega)$ in the definition of $U^e(\Omega)$ is due to Colombeau ([10], 1.2.1). We decided to keep the original form of $\mathcal{G}^e(\Omega)$ although all the results of this section would remain valid (and, in fact, even slightly easier to formulate) choosing also $U^e(\Omega)$ to be $T^{-1}(\mathcal{A}_0(\Omega) \times \Omega)$.

15.1 Remark. (i) If we had chosen to set $U^e(\Omega) = T^{-1}(\mathcal{A}_0(\Omega) \times \Omega)$ (contrary to [10], cf. the footnote to (1) above) j would be the identity map on $U^d(\Omega) = U^e(\Omega)$ and Φ_0 would be set-theoretic inclusion, hence injective.

(ii) Regarding the question of injectivity of Φ, the fact that $\mathcal{A}_1(\Omega)$ has been used in [10] and in (1) above to define $U^e(\Omega)$ (as compared to $\mathcal{A}_0(\Omega)$ in [28] for defining $U^d(\Omega)$) is completely irrelevant: Although this choice renders Φ_0 non-injective in general (consider $(0 \neq) R \in \mathcal{C}^\infty(U^d(\mathbb{R})) = \mathcal{C}^\infty(\mathcal{A}_0(\mathbb{R}) \times \mathbb{R})$ given by $(\varphi, x) \mapsto \int \xi\varphi(\xi)\, d\xi$: $\Phi_0 R = 0$ by the very definition of $U^e(\mathbb{R}) = \mathcal{A}_1(\mathbb{R}) \times \mathbb{R}$), $\mathcal{M} := \ker \Phi_0$ is contained in $\mathcal{N}^d$ since each $R \in \ker \Phi_0$ vanishes identically on pairs $(\phi(\varepsilon, x), x)$ where ϕ is a test object taking values in $\mathcal{A}_q(\mathbb{R})$ ($q \geq 1$). Thus the canonical image of $\mathcal{M}$ in $\mathcal{G}^d := \mathcal{E}^d_M / \mathcal{N}^d$ is trivial.

Having discussed Φ in detail, we will omit j and Φ_0 from our notation in the sequel. Now we can state precisely which properties a function $R : U^d(\Omega) \to \mathbb{C}$ has to satisfy if it is to refute the injectivity of Φ:

(i) $R \in \mathcal{E}^d$, i.e., R has to be smooth;

(ii) $R \in \mathcal{E}^d_M$,

(iii) $R \notin \mathcal{N}^d$,

(iv) $R \in \mathcal{N}^e$.

In the following, we will define maps $P, Q : U^d(\mathbb{R}) \to \mathbb{C}$ each of which satisfies (i)–(iv) above, thereby providing a counterexample to the conjecture of the canonical map Φ being injective. We will give the complete argument for P while only indicating how to adapt the proof to get the analogous result for Q.

For the definition of P, Q let $s := 1$, $\Omega := \mathbb{R}$. We continue using the C-formalism. Although now $U(\Omega) = \mathcal{A}_0(\mathbb{R}) \times \mathbb{R} = \mathcal{A}_0(\Omega) \times \Omega$ note that the C-formalism, nevertheless, differs from the J-formalism with respect to embedding $\mathcal{D}'$, differentiation, testing (which involves T in the case of the J-formalism) and, finally, with respect to the action induced by a diffeomorphism. As a prerequisite for writing down P, Q

explicitly, we introduce the following notation:

$$
\begin{array}{lrcll}
 & \langle\varphi|\varphi\rangle & := & \int \varphi(\xi)\overline{\varphi(\xi)}\,d\xi & (\varphi \in \mathcal{D}(\mathbb{R})) \\
v_k \in \mathcal{D}'(\mathbb{R}): & \langle v_k, \varphi\rangle & := & \int \xi^k \varphi(\xi)\,d\xi & (\varphi \in \mathcal{D}(\mathbb{R}),\ k \in \mathbb{N}_0) \\
v_{\frac{1}{2}} \in \mathcal{D}'(\mathbb{R}): & \langle v_{\frac{1}{2}}, \varphi\rangle & := & \int |\xi|^{\frac{1}{2}} \varphi(\xi)\,d\xi & (\varphi \in \mathcal{D}(\mathbb{R})) \\
 & v(\varphi) & := & \langle\varphi|\varphi\rangle^{\frac{1}{2}} \langle v_{\frac{1}{2}}, \varphi\rangle & (\varphi \in \mathcal{D}(\mathbb{R})) \\
 & g(x) & := & \frac{x}{1+x^2} & (x \in \mathbb{R}) \\
 & e(x) & := & \begin{cases} \exp(-\frac{1}{x}) & (x > 0) \\ 0 & (x \le 0) \end{cases} & (x \in \mathbb{R}) \\
 & \gamma_k & := & k + \frac{1}{k} & (k \in \mathbb{N}).
\end{array}
$$

Finally, choose an (even) function $\sigma \in \mathcal{D}(\mathbb{R})$ satisfying $0 \le \sigma \le 1$, $\sigma(x) \equiv 1$ for $|x| \le \frac{1}{2}$, $\sigma(x) \equiv 0$ for $|x| \ge \frac{3}{2}$ and set

$$
h_k(x) := \sigma(x) \cdot 2g(x) + (1 - \sigma(x)) \cdot \operatorname{sgn}(x) \cdot |2g(x)|^{\gamma_k} \qquad (x \in \mathbb{R},\ k \in \mathbb{N}).
$$

Being bounded and linear resp. bilinear (over $\mathbb{R}$), v_k, $v_{\frac{1}{2}}$ and $\langle\,.\,|\,.\,\rangle$ are smooth on $\mathcal{D}(\mathbb{R})$ ($k \in \mathbb{N}_0$). On $\mathcal{A}_0(\mathbb{R})$, $\langle\varphi|\varphi\rangle > 0$. Thus v is smooth on $\mathcal{A}_0(\mathbb{R})$ as a product of smooth functions. Observe that $\mathcal{A}_q(\mathbb{R}) = \mathcal{A}_0(\mathbb{R}) \cap \bigcap_{k=1}^{q} \ker v_k$.

In the sequel, we will make use of the following facts concerning g and e: For every $n \in \mathbb{N}_0$ there exists a constant $c_n > 0$ such that for all $x \ne 0$

$$
|g^{(n)}(x)| \le \frac{c_n}{|x|^{n+1}}.
$$

The derivatives of e can be written in the following form:

$$
e^{(n)}(x) = e(x) \cdot \frac{q_n(x)}{x^{2r}} = \begin{cases} \exp(-\frac{1}{x}) \cdot \frac{q_n(x)}{x^{2r}} & (x > 0) \\ 0 & (x \le 0) \end{cases}
$$

for every $n \in \mathbb{N}$ where q_n is a polynomial of degree $n - 1$ and $\frac{0}{0} := 0$.

Scaling of φ produces the following relations:

$$
\begin{array}{rcl}
\langle S_\varepsilon\varphi|S_\varepsilon\varphi\rangle & = & \frac{1}{\varepsilon}\langle\varphi|\varphi\rangle \\
\langle v_k, S_\varepsilon\varphi\rangle & = & \varepsilon^k \langle v_k, \varphi\rangle \\
\langle v_{\frac{1}{2}}, S_\varepsilon\varphi\rangle & = & \varepsilon^{\frac{1}{2}} \langle v_{\frac{1}{2}}, \varphi\rangle \\
v(S_\varepsilon\varphi) & = & v(\varphi).
\end{array}
$$

Apart from abbreviating $R \circ S^{(\varepsilon)} = R \circ (S_\varepsilon \times \mathrm{id})$ as R_ε for any function R defined on $\mathcal{A}_0(\mathbb{R}) \times \mathbb{R}$, we also will write R_ε for $R \circ S_\varepsilon$ if R is defined on $\mathcal{A}_0(\mathbb{R})$.

15.2 Definition. *Let $\varphi \in \mathcal{A}_0(\mathbb{R})$, $x \in \mathbb{R}$ and set*

$$\begin{aligned} P(\varphi, x) &:= \sum_{k=1}^{\infty} \frac{1}{k!} \cdot g\big(\langle\varphi|\varphi\rangle^{\gamma_k} e(v(\varphi))\big) \cdot \langle\varphi|\varphi\rangle^{\gamma_k} \cdot \langle v_k, \varphi\rangle, \qquad (3)\\ Q(\varphi, x) &:= \sum_{k=1}^{\infty} \frac{1}{k!} \cdot h_k\big(\langle\varphi|\varphi\rangle^{\frac{3}{2}} \langle v_{\frac{1}{2}}, \varphi\rangle\big) \cdot \langle\varphi|\varphi\rangle^{\gamma_k} \cdot \langle v_k, \varphi\rangle. \qquad (4)\end{aligned}$$

Hence P and Q, in fact, only depend on φ. We will see below that both series for P and Q converge uniformly on bounded subsets of $\mathcal{A}_0(\mathbb{R})$, making P and Q well-defined. For $k \in \mathbb{N}$, $\varphi \in \mathcal{A}_0(\mathbb{R})$ set

$$P_k(\varphi) := g\big(\langle\varphi|\varphi\rangle^{\gamma_k} e(v(\varphi))\big) \cdot \langle\varphi|\varphi\rangle^{\gamma_k} \cdot \langle v_k, \varphi\rangle.$$

Fix a positive number $\eta \leq 1$. To establish properties (i) and (ii) (i.e., smoothness and moderateness) of P, we will derive estimates of the form

$$|(\mathrm{d}^n(P_k)_\varepsilon)(\varphi)(\psi_1, \ldots, \psi_n)| \leq C_n^k \cdot \varepsilon^{-\frac{1}{k} - n\eta} \qquad (5)$$

for some constants $C_n \geq 1$ not depending on ε ($n \in \mathbb{N}_0$), uniformly on any bounded subset B of $\mathcal{D}(\mathbb{R})$ and $\varphi \in B \cap \mathcal{A}_0(\mathbb{R})$, $\psi_1, \ldots, \psi_n \in B \cap \mathcal{A}_{00}(\mathbb{R})$. Setting $\varepsilon = 1$ in (5) shows that for each $n \in \mathbb{N}_0$, $(\mathrm{d}^n P_k)$ is an (eb)-sequence on $\mathcal{A}_0(\mathbb{R}) \times \mathcal{A}_{00}(\mathbb{R})^n$ which, by Theorem 14.4, implies smoothness of P. Considering arbitrary values of $\varepsilon \in I$, on the other hand, will lead to the proof of moderateness of P.

15.1 Proof of the estimates (5)

Fix $n \in \mathbb{N}_0$, $\varepsilon \in I$ and $0 < \eta \leq 1$. Set

$$\begin{aligned} P_k^{(1)}(\varphi) &:= g\big(\langle\varphi|\varphi\rangle^{\gamma_k} e(v(\varphi))\big)\\ P_k^{(2)}(\varphi) &:= \langle\varphi|\varphi\rangle^{\gamma_k}\\ P_k^{(3)}(\varphi) &:= \langle v_k, \varphi\rangle.\end{aligned}$$

(5) is equivalent to saying that the functions $\varepsilon^{\frac{1}{k}+n\eta} \cdot \mathrm{d}^n(P_k)_\varepsilon$ form an (eb)-sequence, with the respective constants in the estimate independent of ε. In order to prove this, by Leibniz' rule for the differential of a product and by Proposition 14.3 it suffices to show that each of the sequences (indexed by $k \in \mathbb{N}$) $\varepsilon^{n\eta}\mathrm{d}^m(P_k^{(1)})_\varepsilon$, $\varepsilon^{\gamma_k}\mathrm{d}^m(P_k^{(2)})_\varepsilon =$

$\mathrm{d}^m(P_k^{(2)})$ and $\varepsilon^{-k}\mathrm{d}^m(P_k^{(3)})_\varepsilon = \mathrm{d}^m(P_k^{(3)})$ is (eb) for $m \le n$, independently of ε. In the following, we are going to verify these claims step by step, starting with the elementary building blocks of the series defining P resp. Q.

Remark. For $w \in \mathcal{D}'(\Omega)$, $(\mathrm{d}w)(\varphi)(\psi) = \langle w, \psi\rangle$ and $\mathrm{d}^l w = 0$ for $l \ge 2$, due to the linearity of w. $\langle\,.\,|\,.\,\rangle$ being bilinear over $\mathbb{R}$, we obtain $(\mathrm{d}\langle\,.\,|\,.\,\rangle)(\varphi)(\psi) = \langle\psi|\varphi\rangle + \langle\varphi|\psi\rangle$, $(\mathrm{d}^2\langle\,.\,|\,.\,\rangle)(\varphi)(\psi_1,\psi_2) = \langle\psi_1|\psi_2\rangle + \langle\psi_2|\psi_1\rangle$ and $\mathrm{d}^l\langle\,.\,|\,.\,\rangle = 0$ for $l \ge 3$.

15.3 Proposition. *The following sequences of functions of φ (indexed by $k \in \mathbb{N}$) are* (eb) *(1. and 3. on $\mathcal{D}(\mathbb{R})$, 2. on $\mathcal{A}_0(\mathbb{R})$):*

1. $\langle\xi^k, \varphi(\xi)\rangle$, $\quad\langle|\xi|^k, \varphi(\xi)\rangle$, $\quad\langle\varphi|\varphi\rangle^k$, $\quad\langle\varphi|\varphi\rangle^{\gamma_k}$;
2. $\langle\varphi|\varphi\rangle^{-k}$, $\quad\langle\varphi|\varphi\rangle^{-\gamma_k}$;
3. $(\beta_k)_n := \beta_k(\beta_k - 1)\dots(\beta_k - n + 1) \qquad$ *(for fixed $n \in \mathbb{N}_0$; $(\beta_k)_0 := 1$)*

where the numbers $\beta_k \in \mathbb{R}$ occurring in 3. only have to satisfy an estimate of the form $|\beta_k| \le pk$ for some fixed $p \in \mathbb{N}$.

Proof. Fix a bounded subset B of $\mathcal{D}(\mathbb{R})$ containing at least one $\varphi \neq 0$. Then there exists a bounded set $L \subseteq \mathbb{R}$ containing the supports of all $\varphi \in B$. Let $\mathrm{m}(L) > 0$ denote the Lebesgue measure of L and set $C_1 := \max(1, \sup\limits_{\xi\in L}|\xi|)$, $C_2 := \max(1, \mathrm{m}(L))$. Moreover, $C_3 := \max(1, \sup\limits_{\varphi\in B}\|\varphi\|_\infty)$ is finite. Now let $\varphi \in B$.

1. We have

$$\max\left(|\langle\xi^k, \varphi(\xi)\rangle|\,,\,|\langle|\xi|^k, \varphi(\xi)\rangle|\right) \le C_1^k C_2 C_3 \le (C_1C_2C_3)^k,$$

$$\langle\varphi|\varphi\rangle^k \le (C_2C_3^2)^k,$$
$$\langle\varphi|\varphi\rangle^{\gamma_k} \le (C_2C_3^2)^{k+1} \le (C_2^2C_3^4)^k.$$

2. The Schwarz inequality yielding $1 = \langle 1, \varphi\rangle \le (\int\limits_L 1)^{\frac{1}{2}}\|\varphi\|_2 = (\mathrm{m}(L)\langle\varphi|\varphi\rangle)^{\frac{1}{2}}$, we conclude $\langle\varphi|\varphi\rangle \ge C_2^{-1}$ and from this, in turn,

$$\langle\varphi|\varphi\rangle^{-k} \le C_2^k,$$

$$\langle\varphi|\varphi\rangle^{-\gamma_k} \le (C_2^2)^k.$$

3. The case $n = 0$ being trivial, note that there exists $C_0 > 1$ such that $k^n \le C_0^k$ for all $k \in \mathbb{N}$. Consequently,

$$|(\beta_k)_n| \le (|\beta_k| + n - 1)^n \le (pk + n - 1)^n \le (pkn)^n \le (C_0C)^k$$

where $C := \max(1, (pn)^n)$; the third inequality in the preceding chain is based on $0 \le (pk-1)(n-1)$. $\square$

Now the (ep)-property for $\varepsilon^{-k}\mathrm{d}^m(P_k^{(3)})_\varepsilon = \mathrm{d}^m(P_k^{(3)}) = \mathrm{d}^m(\langle v_k, .\,\rangle)$ is clear from Proposition 15.3 and the remark preceding it. Regarding $\varepsilon^{\gamma_k}\mathrm{d}^m(P_k^{(2)})_\varepsilon = \mathrm{d}^m(P_k^{(2)}) = \mathrm{d}^m(\langle\,.\,|\,.\,\rangle^{\gamma_k})$ we obtain from the chain rule that $(\mathrm{d}^m\langle\,.\,|\,.\,\rangle^{\gamma_k})(\varphi)(\psi_1,\dots,\psi_m)$ is given as a certain sum of terms of the form

$$(\gamma_k)_l\langle\varphi|\varphi\rangle^{\gamma_k-l}\cdot(\mathrm{d}^{j_1}\langle\,.\,|\,.\,\rangle)(\dots)\cdot\ldots\ldots\cdot\mathrm{d}^{j_l}\langle\,.\,|\,.\,\rangle)(\dots), \tag{6}$$

the groups of three dots in parentheses standing for φ and certain subsequences of $(\psi_1,\dots,\psi_m)$. ($j_1+\dots+j_l = m$ and, for any non-vanishing term of the above form, $j_1,\dots,j_l \in \{1,2\}$.) (6) immediately allows the application of Propositions 14.3 and 15.3, again in connection with the remark preceding the latter, thereby establishing the (eb)-property also for $\varepsilon^{\gamma_k}\mathrm{d}^m(P_k^{(2)})_\varepsilon$. For both terms treated so far, the constants occurring in the (eb)-estimate obviously are independent of ε. Observe that the case $n=0$ of (5) is already settled completely on the basis of the results obtained so far, due to g being globally bounded on $\mathbb{R}$.

We now turn to the remaining one of the three terms which, to be sure, is the most difficult one to handle: We have to show $(\varepsilon^{n\eta}\mathrm{d}^m(P_k^{(1)})_\varepsilon)_k$ to constitute an (eb)-sequence, with the corresponding constant not depending on ε. Again according to the chain rule, the m-th differential of $\varepsilon^{n\eta}g\big(\varepsilon^{-\gamma_k}\langle\varphi|\varphi\rangle^{\gamma_k}e(v(\varphi))\big)$, evaluated at $\varphi;\psi_1,\dots,\psi_m$, is given as a sum of terms of the form

$$\varepsilon^{n\eta-l\gamma_k}g^{(l)}\big(\varepsilon^{-\gamma_k}\langle\varphi|\varphi\rangle^{\gamma_k}e(v(\varphi))\big)\cdot f_1(\dots)\cdot\ldots\ldots\cdot f_l(\dots) \qquad (1\le l\le m)$$

(maintaining the convention that a group of three dots at a differential's argument's place always denotes a certain subsequence of $(\varphi;\psi_1,\dots,\psi_l)$) where each $f_{l'}(\dots)$ is of the form

$$\mathrm{d}^j\big(\langle\,.\,|\,.\,\rangle^{\gamma_k}\cdot(e\circ v)\big)(\dots) \qquad (1\le j\le l\le m). \tag{7}$$

On the basis of Leibniz' rule, Proposition 14.3 and the fact that $\langle\varphi|\varphi\rangle^{\gamma_k}$ together with all its differentials is already known to be (eb), it will suffice to deal with terms of the form

$$\varepsilon^{n\eta-l\gamma_k}g^{(l)}\big(\varepsilon^{-\gamma_k}\langle\varphi|\varphi\rangle^{\gamma_k}e(v(\varphi))\big)\cdot\mathrm{d}^{i_1}(e\circ v)(\dots)\cdot\ldots\ldots\cdot\mathrm{d}^{i_l}(e\circ v)(\dots) \tag{8}$$

where $0\le i_{l'}\le l$ and $i_1+\dots i_l\le m$. To this end, we have to analyze $\mathrm{d}^i(e\circ v)$ for $0\le i\le j\le l\le m$. Once more by the chain rule, this is a sum of products of $e^{(r)}(v(\varphi))$ with r factors which are differentials of v $(0\le r\le i)$. The proof of Proposition 15.3 shows that $\langle\varphi|\varphi\rangle^{\frac{1}{2}-r'}$ is bounded on bounded sets for $r'\in\mathbb{N}$. From

this it follows that also the differentials of v are bounded on bounded sets; as they do not depend on k, they form an (eb)-sequence in a trivial manner. By Proposition 14.3 again we can discard them for the rest of the argument. Thus we are left with estimating

$$\varepsilon^{n\eta-l\gamma_k} g^{(l)}\big(\varepsilon^{-\gamma_k}\langle\varphi|\varphi\rangle^{\gamma_k} e(v(\varphi))\big)\cdot e^{(r_1)}(v(\varphi))\cdot\dots\cdot e^{(r_l)}(v(\varphi)) \qquad (1\le l\le m) \quad (9)$$

where $0 \le r_t \le i_t \le l$ $(1 \le t \le l)$ and $r_1+\dots+r_l \le m$. Now $e^{(r)}(v(\varphi))$ can be written as

$$e^{(r)}(v(\varphi)) = e(v(\varphi))\frac{q_r(v(\varphi))}{v(\varphi)^{2r}}$$

where q_r is a certain polynomial of degree $r-1$. Consequently, (9) takes the form

$$\varepsilon^{n\eta-l\gamma_k} g^{(l)}(X)\cdot e(v(\varphi))^l \cdot \frac{1}{v(\varphi)^{2n}}\cdot v(\varphi)^{2(n-\bar r)}\prod_{t=1}^{l} q_{r_t}(v(\varphi))$$

where we have set $X := \varepsilon^{-\gamma_k}\langle\varphi|\varphi\rangle^{\gamma_k} e(v(\varphi))$ and $\bar r := \sum_{t=1}^{l} r_t$, for the sake of brevity. Now expand $e(v(\varphi))^l$ according to

$$e(v(\varphi))^l = X^{l(1-\frac{\eta}{\gamma_k})}\cdot\left(\varepsilon^{\gamma_k}\langle\varphi|\varphi\rangle^{-\gamma_k}\right)^{l(1-\frac{\eta}{\gamma_k})}\cdot e(v(\varphi))^{l\frac{\eta}{\gamma_k}}$$

and regroup the terms in the following way as to obtain the desired estimates:

1. Collecting all powers of ε, we obtain $\varepsilon^{n\eta-l\gamma_k}\cdot(\varepsilon^{\gamma_k})^{l(1-\frac{\eta}{\gamma_k})} = \varepsilon^{(n-l)\eta} \le 1$.

2. For $|X| \le 1$, we have $\left|g^{(l)}(X)\cdot X^{l(1-\frac{\eta}{\gamma_k})}\right| \le \left|g^{(l)}(X)\right| \le \|g^{(l)}\|_\infty$ (note that $0 < \eta \le 1 < \gamma_1 = 1+1 \le \gamma_k$ and that, consequently, $\frac{\eta}{\gamma_k} \le \frac{1}{2}$ for all $k \in \mathbb{N}$), while for $|X| \ge 1$ and c_l denoting a positive constant dominating $|x|^{l+1}|g^{(l)}(x)|$ for all $x \in \mathbb{R}$ (see the remarks after the introduction of g), we obtain

$$\left|g^{(l)}(X)\cdot X^{l(1-\frac{\eta}{\gamma_k})}\right| \le c_l\,|X|^{-l-1+l-l\frac{\eta}{\gamma_k}} \le c_l.$$

Altogether, the function $X \mapsto g^{(l)}(X)\cdot X^{l(1-\frac{\eta}{\gamma_k})}$ $(l = 1,\dots,n)$ is globally bounded by a positive constant larger or equal to 1, say, C_g.

3. The following term, that is $\langle\varphi|\varphi\rangle^{-\gamma_k l(1-\frac{\eta}{\gamma_k})} = \langle\varphi|\varphi\rangle^{-l(\gamma_k-\eta)}$ gives rise to an (eb)-sequence letting $k = 1,2,\dots$: This is immediate from $\langle\varphi|\varphi\rangle^{-(\gamma_k-\eta)} = \langle\varphi|\varphi\rangle^{-\gamma_k}\cdot\langle\varphi|\varphi\rangle^{\eta}$ and Propositions 15.3 and 14.3, together with the observation that $\langle\varphi|\varphi\rangle^{\eta}$ is bounded on bounded sets. Hence for a given bounded subset B of $\mathcal{A}_0(\mathbb{R})$ there exists a constant $C_1 \ge 1$ satisfying $\langle\varphi|\varphi\rangle^{-l(\gamma_k-\eta)} \le C_1^k$ for all $\varphi \in B$, $k \in \mathbb{N}$.

4. $e(v(\varphi))^{l\frac{\eta}{\gamma_k}} \cdot \frac{1}{v(\varphi)^{2n}}$ can be rewritten as

$$e\left(\frac{\gamma_k v(\varphi)}{l\eta}\right) \cdot \left(\frac{l\eta}{\gamma_k v(\varphi)}\right)^{2n} \cdot \frac{1}{(l\eta)^{2n}} \cdot \gamma_k^{2n}.$$

Now $e(x) \cdot x^{-2n}$ (with $\frac{0}{0} := 0$) is globally bounded on $\mathbb{R}$ and $\gamma_k^{2n} \leq (k+1)^{2n} \leq (2k)^{2n} = 4^n k^{2n}$; the latter is (eb) by the proof of part 3 of Proposition 15.3. Therefore, $e(v(\varphi))^{l\frac{\eta}{\gamma_k}} \cdot \frac{1}{v(\varphi)^{2n}}$ can be estimated by C_2^k for a suitable constant $C_2 \geq 1$.

5. φ ranging over the bounded set B as in 3. above, $v(\varphi)$ attains values in a bounded subset of $\mathbb{C}$. On this set the polynomial $x^{2(n-\bar{r})} \prod_{t=1}^{l} q_{r_t}(x)$ is bounded by some constant $C_3 \geq 1$.

Summarizing, for any given bounded subset B of $\mathcal{A}_0(\mathbb{R})$ there exist constants C_g, C_1, C_2, C_3 (independent of $\varepsilon \in I$) such that

$$\left|\varepsilon^{n\eta - l\gamma_k} g^{(l)}\left(\varepsilon^{-\gamma_k} \langle\varphi|\varphi\rangle^{\gamma_k} e(v(\varphi))\right) \cdot e^{(r_1)}(v(\varphi)) \cdot \dots \cdot e^{(r_l)}(v(\varphi))\right| \leq (C_g C_1 C_2 C_3)^k$$

for all $\varphi \in B$. This completes the proof of (5). □

15.2 Proof of smoothness of P

Setting $\varepsilon := 1$ in (5) shows $(\mathrm{d}^n P_k)$ to be an (eb)-sequence on $\mathcal{A}_0(\mathbb{R}) \times \mathcal{A}_{00}(\mathbb{R})^n$, for each $n \in \mathbb{N}_0$. Theorem 14.4 now implies that P as defined in 15.2 is smooth, that the differentials of P can be computed term-wise and that all the series for $\mathrm{d}^n P$ ($n \in \mathbb{N}_0$) converge with respect to τ_∞. □

15.3 Proof of moderateness of P

Let B be a bounded subset of $\mathcal{D}(\mathbb{R})$ and assume C_n ($n \in \mathbb{N}_0$) to be appropriate constants as to satisfy (5) for all $\varphi \in B \cap \mathcal{A}_0(\mathbb{R})$, $\psi_1, \dots, \psi_n \in B \cap \mathcal{A}_{00}(\mathbb{R})$. Choosing $\eta := 1$, say, estimate (5) results in $|(\mathrm{d}^n (P_k)_\varepsilon)(\varphi)(\psi_1, \dots, \psi_n)| \leq C_n^k \cdot \varepsilon^{-\frac{1}{k} - n}$. Multiplying by $\frac{1}{k!}$ and forming the infinite sum constituting $\mathrm{d}^n P_\varepsilon$, we obtain

$$|(\mathrm{d}^n P_\varepsilon)(\varphi)(\psi_1, \dots, \psi_n)| \leq (e^{C_n} - 1) \cdot \varepsilon^{-1-n},$$

uniformly on B in the sense specified above. Hence P satisfies the condition equivalent to moderateness given in Theorem 7.12. □

We proceed to prove $P \notin \mathcal{N}^d$ resp. $P \in \mathcal{N}^e$. For the former negligibility property, instead of the condition as given in 7.3, we use the equivalent condition (once again

omitting the quantifiers "$\forall K \subset\subset \Omega\ \forall \alpha \in \mathbb{N}_0^s$")

$$R \in \mathcal{N}^d \Leftrightarrow \forall n\ \exists q\ \forall B\,(\text{bounded}) \subseteq \mathcal{A}_q(\mathbb{R}^s): \sup_{x\in K,\ \varphi\in B} |\partial^\alpha(R(S_\varepsilon\varphi, x))| = O(\varepsilon^n)$$

occurring as 1° in Theorem 18 of [28]. Observe that for the application of the latter theorem, we need the fact that $P \in \mathcal{E}_M^d$ which has been shown above. For $\mathcal{N}^e$ we use the modified defining condition as given previously in this section (cf. the discussion of Φ_0):

$$R \in \mathcal{N}^e \Leftrightarrow \forall n\ \exists q \qquad\qquad \forall \varphi \in \mathcal{A}_q(\mathbb{R}^s): \quad \sup_{x\in K} |\partial^\alpha(R(S_\varepsilon\varphi, x))| = O(\varepsilon^n)$$

Clearly, our choice of the above forms of the respective conditions is motivated by the intention to have them as similar as possible to highlight the essential difference between them: The estimate on $|\partial^\alpha(R^e(S_\varepsilon\varphi, x))|$ is required to hold uniformly on bounded subsets with respect to φ in the former case as compared to only pointwise in the latter.

15.4 Proof of $P \notin \mathcal{N}^d$

Set $K := \{0\}$, $\alpha := 0$, $n := 1$. We are going to show that for this set of data the condition for P to belong to $\mathcal{N}^d$ is violated, i.e., we are going to show that for every $q \in \mathbb{N}$ there exists a bounded subset B of $\mathcal{A}_q(\mathbb{R})$ such that $\sup_{\varphi\in B} |(P(S_\varepsilon\varphi, 0))|$ is *not* of order $O(\varepsilon)$. To this end, let $q \in \mathbb{N}$. Since $v_{\frac{1}{2}}, v_0, v_1, \dots, v_{q+1}$ are linearly independent in $\mathcal{D}'(\mathbb{R})$ there exist $\varphi_0, \varphi_1 \in \mathcal{A}_q(\mathbb{R})$ satisfying

$$\begin{aligned} \langle v_{\frac{1}{2}}, \varphi_0\rangle &= 0, \qquad & \langle v_{q+1}, \varphi_0\rangle &= 1,\\ \langle v_{\frac{1}{2}}, \varphi_1\rangle &= 1, \qquad & \langle v_{q+1}, \varphi_1\rangle &= 1. \end{aligned}$$

Setting $\varphi_\lambda := (1-\lambda)\varphi_0 + \lambda\varphi_1$ $(0 \le \lambda \le 1)$, $B := \{\varphi_\lambda \mid 0 \le \lambda \le 1\}$ is a bounded subset of $\mathcal{A}_q(\mathbb{R})$; moreover, $\langle v_{\frac{1}{2}}, \varphi_\lambda\rangle = \lambda$. For each λ in a suitable interval $(0, \lambda_0]$ we are going to specify some $\varepsilon_\lambda \in I$ with $\varepsilon_\lambda \to 0$ as $\lambda \to 0$ such that $P_{\varepsilon_\lambda}(\varphi_\lambda, 0) \to \infty$ $(\lambda \to 0)$. Consequently, $\sup_{\varphi\in B} |(P(S_\varepsilon\varphi, 0))|$ is not even of order $O(1)$, i.e. not even bounded as $\varepsilon \to 0$. The (nonnegative) function defined by the assignment

$$\lambda \mapsto \varepsilon_\lambda := \langle \varphi_\lambda | \varphi_\lambda\rangle \cdot e(v(\varphi_\lambda))^{\frac{1}{\gamma_{q+1}}}$$

is continuous for $\lambda \in [0,1]$, strictly positive for $\lambda > 0$ and satisfies $\varepsilon_0 = 0$. Hence there exists $\lambda_0 > 0$ such that $\varepsilon_\lambda \in I$ for $0 \le \lambda \le \lambda_0$. Moreover, $\varepsilon_\lambda \to 0$ as $\lambda \to 0$.

The general term of the series defining $P_\varepsilon(\varphi_\lambda, 0)$ is given by (apart from the factor $\frac{1}{k!}$)

$$(P_k)_\varepsilon(\varphi_\lambda) = \varepsilon^{-\frac{1}{k}} \cdot g\big(\varepsilon^{-\gamma_k}\langle\varphi_\lambda|\varphi_\lambda\rangle^{\gamma_k} e(v(\varphi_\lambda))\big) \cdot \langle\varphi_\lambda|\varphi_\lambda\rangle^{\gamma_k} \cdot \langle v_k, \varphi_\lambda\rangle.$$

For $k = 1, \ldots, q$ this expression vanishes identically on B due to $\varphi_\lambda \in \mathcal{A}_q(\mathbb{R})$. For $k \geq q+2$ it can be estimated by $\varepsilon^{-\frac{1}{k}} \cdot \frac{1}{2} \cdot C^k$ (note that $\|g\|_\infty = \frac{1}{2}$) for some constant $C \geq 1$ being independent of λ since $\langle\varphi|\varphi\rangle^{\gamma_k}\langle v_k, \varphi\rangle$ forms an (eb)-sequence. Consequently,

$$\sum_{k=q+2}^{\infty} \frac{1}{k!}(P_k)_\varepsilon(\varphi_\lambda) \leq \varepsilon^{-\frac{1}{q+2}} \cdot \frac{1}{2} \cdot e^C.$$

It remains to look at the leading term, that is, $(P_{q+1})_\varepsilon(\varphi_\lambda)$ (again omitting $\frac{1}{(q+1)!}$). Setting $\varepsilon := \varepsilon_\lambda$ it takes the value

$$\varepsilon_\lambda^{-\frac{1}{q+1}} \cdot g(1) \cdot \langle\varphi_\lambda|\varphi_\lambda\rangle^{\gamma_{q+1}} \cdot 1 \;=\; \varepsilon_\lambda^{-\frac{1}{q+1}} \cdot \frac{1}{2} \cdot \langle\varphi_\lambda|\varphi_\lambda\rangle^{\gamma_{q+1}}.$$

Altogether we obtain

$$\begin{aligned} P_{\varepsilon_\lambda}(\varphi_\lambda, 0) &\geq \frac{1}{(q+1)!} \cdot \varepsilon_\lambda^{-\frac{1}{q+1}} \cdot \frac{1}{2} \cdot \langle\varphi_\lambda|\varphi_\lambda\rangle^{\gamma_{q+1}} - \varepsilon^{-\frac{1}{q+2}} \cdot \frac{1}{2} \cdot e^C \\ &= \varepsilon_\lambda^{-\frac{1}{q+1}} \cdot \frac{1}{2}\left[\frac{\langle\varphi_\lambda|\varphi_\lambda\rangle^{\gamma_{q+1}}}{(q+1)!} - \varepsilon_\lambda^{\frac{1}{(q+1)(q+2)}} \cdot e^C\right] \end{aligned}$$

which tends to infinity as $\lambda \to 0$ (and, consequently, $\varepsilon_\lambda \to 0$), due to $\langle\varphi_\lambda|\varphi_\lambda\rangle$ being bounded from below uniformly for $\lambda \in [0,1]$ and the second term in the square bracket vanishing in the limit. $\square$

15.5 Proof of $P \in \mathcal{N}^e$

Let $K \subset\subset \mathbb{R}$, $\alpha := 0$ (note that $P_\varepsilon(\varphi, x)$ does not depend on x) and $n \in \mathbb{N}$ be given; we claim that $q := n-1$ is an appropriate choice for showing that

$$\forall \varphi \in \mathcal{A}_q(\mathbb{R}) : \sup_{x \in K} |P_\varepsilon(\varphi, x)| = O(\varepsilon^n).$$

Let $\varphi \in \mathcal{A}_q(\mathbb{R}) = \mathcal{A}_{n-1}(\mathbb{R})$. If $\langle v_{\frac{1}{2}}, \varphi\rangle \leq 0$ then $v(\varphi) \leq 0$ and, consequently, $e(v(\varphi)) = 0$ which in turn implies $P_\varepsilon(\varphi, x)=0$ for all $x \in \mathbb{R}$ and all $\varepsilon \in I$. Thus we may assume that $\langle v_{\frac{1}{2}}, \varphi\rangle > 0$. But then also $v(\varphi)$ and, in turn, $e(v(\varphi))$ are positive. Taking into account that $|g(x)| = |\frac{1}{x} \cdot \frac{x^2}{x^2+1}| \leq |\frac{1}{x}|$ for $x \neq 0$ we obtain the following estimate:

$$|g\big(\varepsilon^{-\gamma_k}\langle\varphi|\varphi\rangle^{\gamma_k} e(v(\varphi)\big) \cdot \varepsilon^{-\gamma_k}\langle\varphi|\varphi\rangle^{\gamma_k} \cdot \varepsilon^k \langle v_k, \varphi\rangle| \;\leq\; \varepsilon^k \frac{|\langle v_k, \varphi\rangle|}{e(v(\varphi))}.$$

Choosing a constant C satisfying $|\langle v_k, \varphi\rangle| \le C^k$ for all $k \in \mathbb{N}$ (note that $(v_k)_k$ is (eb) by Proposition 15.3) we finally arrive at

$$|P_\varepsilon(\varphi, x)| \le \sum_{k=q+1}^{\infty} \frac{1}{k!} \cdot \frac{\varepsilon^k C^k}{e(v(\varphi))} \le \varepsilon^{q+1} \cdot \frac{e^C}{e(v(\varphi))}$$

thereby completing the proof of $P \in \mathcal{N}^e$. □

Now we turn to briefly discussing Q. In what follows we will tacitly assume all (eb)-questions to be handled appropriately. After scaling φ and dropping the factor $\frac{1}{k!}$ the typical term of the series defining Q takes the form

$$\varepsilon^{-\frac{1}{k}} \cdot h_k\big(\frac{1}{\varepsilon}\langle\varphi|\varphi\rangle\, v(\varphi)\big) \cdot \langle\varphi|\varphi\rangle^{\gamma_k} \cdot \langle v_k, \varphi\rangle.$$

As with P, $\mathrm{d}^m(\langle\,.\,|\,.\,\rangle^{\gamma_k})$ and $\mathrm{d}^m(\langle v_k,\,.\,\rangle)$ are (eb) for all $m \in \mathbb{N}_0$. Modulo some (eb)-arguments again, the non-trivial part of dealing with $\mathrm{d}^m h_k\big(\frac{1}{\varepsilon}\langle\varphi|\varphi\rangle\, v(\varphi)\big)$ consists in getting to grips with $\varepsilon^{-l} h_k^{(l)}\big(\frac{1}{\varepsilon}\langle\varphi|\varphi\rangle\, v(\varphi)\big)$ for $l \le m$. Thanks to the harmless leading factor ε^{-l} (as compared to $\varepsilon^{-l\gamma_k}$ in the case of P) it is sufficient to note that there exists some constant $C \ge 1$ satisfying $\|h_k^{(l)}\|_\infty \le C^k$ for all $k \in \mathbb{N}$ and $0 \le l \le m$ (observe that σ and g are globally bounded together with all their derivatives). Summarizing, we obtain that for all $m \le n$ the sequences (with respect to k) $\varepsilon^m \mathrm{d}^m h_k\big(\frac{1}{\varepsilon}\langle\varphi|\varphi\rangle\, v(\varphi)\big)$ and, consequently,

$$\varepsilon^n \cdot \mathrm{d}^n\Big(h_k\big(\frac{1}{\varepsilon}\langle\varphi|\varphi\rangle\, v(\varphi)\big) \cdot \langle\varphi|\varphi\rangle^{\gamma_k} \cdot \langle v_k, \varphi\rangle\Big)$$

are (eb), with the respective constants not depending on ε. From this, smoothness and moderateness of Q follow. To obtain the proof of $Q \notin \mathcal{N}^d$ from the proof of $P \notin \mathcal{N}^d$ simply replace the former definition of ε_λ by $\varepsilon_\lambda := \langle\varphi_\lambda|\varphi_\lambda\rangle^{\frac{3}{2}} \cdot \lambda$ and use the fact that $h_k(1) = \|h_k\|_\infty = 1$. Finally, to show that $Q \in \mathcal{N}^e$, fix $\varphi \in \mathcal{A}_q(\mathbb{R})$. The case $\langle v_{\frac{1}{2}}, \varphi\rangle = 0$ being trivial, assume that $\langle v_{\frac{1}{2}}, \varphi\rangle \neq 0$. For $\varepsilon \le \frac{2}{3}\langle\varphi|\varphi\rangle^{\frac{3}{2}}|\langle v_{\frac{1}{2}}, \varphi\rangle|$ we have

$$\Big|h_k\Big(\frac{1}{\varepsilon}\langle\varphi|\varphi\rangle^{\frac{3}{2}}\langle v_{\frac{1}{2}}, \varphi\rangle\Big)\Big| = \Big|2g\Big(\frac{1}{\varepsilon}\langle\varphi|\varphi\rangle^{\frac{3}{2}}\langle v_{\frac{1}{2}}, \varphi\rangle\Big)\Big|^{\gamma_k} \le \varepsilon^{\gamma_k}\Bigg(\frac{2}{\langle\varphi|\varphi\rangle^{\frac{3}{2}}\big|\langle v_{\frac{1}{2}}, \varphi\rangle\big|}\Bigg)^{\gamma_k}.$$

The rest of the argument is similar to that for P.

The reader might ask if it is indeed necessary to come up with counterexamples as complicated as P and Q certainly are. The author doubts that easier ones might be possible. This view is based on reflecting on the rôles each of the three factors constituting a single term of the series (for P, say) in fact has to play:

- $\langle v_k, \varphi\rangle$ distinguishes between the spaces $\mathcal{A}_q(\mathbb{R})$; this is crucial for the negligibility properties.

- $\langle\varphi|\varphi\rangle^{\gamma_k} = \langle\varphi|\varphi\rangle^k \cdot \langle\varphi|\varphi\rangle^{\frac{1}{k}}$, on the one hand, after scaling of φ compensates for the factor ε^k generated by scaling φ in $\langle v_k, \varphi\rangle$. On the other hand, it introduces a factor $\varepsilon^{-\frac{1}{k}}$ making the first non-vanishing term of the series the dominant one as $\varepsilon \to 0$.

- $g(\langle\varphi|\varphi\rangle^{\gamma_k} e(\langle v, \varphi\rangle))$ allows the pointwise vs. uniformly distinction being necessary to obtain $P \notin \mathcal{N}^d$, $P \in \mathcal{N}^e$. Though $g(\langle\varphi|\varphi\rangle^{\gamma_k}\langle v, \varphi\rangle)$ would suffice to achieve the latter, this alternative choice for the argument of g would produce, via the chain rule, a factor $\varepsilon^{-n(k+\frac{1}{k})}$ in the k-th term of $\mathrm{d}^n P_\varepsilon$ which would be disastrous for the moderateness of P. The function e (together with $\varepsilon^{-\gamma_k}$ in the argument of g) suppressing this unwanted factor, P becomes moderate in the end.

Similar arguments apply to Q.

16 Classification of smooth Colombeau algebras between $\mathcal{G}^d(\Omega)$ and $\mathcal{G}^e(\Omega)$

16.1 The development leading from $\mathcal{G}^e(\Omega)$ to $\mathcal{G}^d(\Omega)$

This section, in fact, does justice to the title of the paper by going back to the roots of Colombeau algebras constructed according to the scheme outlined in section 3 of Part I. Surveying the range of algebras lying between the algebras $\mathcal{G}^d(\Omega)$ and (the smooth version of) $\mathcal{G}^e(\Omega)$, we will discuss, in particular, to which extent at least the definition of the algebra $\mathcal{G}^1(\Omega)$ of [13] (which can be located within that range) has to be modified to obtain diffeomorphism invariance. To be sure, the introduction of $\mathcal{G}^1(\Omega)$ has to be considered as the decisive step towards the construction of a diffeomorphism invariant Colombeau algebra. The result of our analysis will be the construction of a diffeomorphism invariant Colombeau algebra $\mathcal{G}^2(\Omega)$ which is non-isomorphic to $\mathcal{G}^d(\Omega)$, yet closer to $\mathcal{G}^1(\Omega)$ than $\mathcal{G}^d(\Omega)$ is.

Apart from $\mathcal{G}^e(\Omega)$, all algebras to be considered in this and the subsequent section have $\mathcal{C}^\infty(U(\Omega))$ resp. $\mathcal{C}^\infty(\mathcal{A}_0(\Omega) \times \Omega)$ as their basic space. In particular, they are smooth algebras in the sense that representatives R have to be smooth also with respect to φ. The maps σ, ι, D_i and the actions induced by a diffeomorphism are defined as in 7.1 and 5.5–5.8, respectively. The algebras will differ, however, as to the type of test objects used for selecting the moderate resp. negligible members

from the basic space. We begin by briefly reviewing the development leading from $\mathcal{G}^e(\Omega)$ via $\mathcal{G}^1(\Omega)$ to $\mathcal{G}^d(\Omega)$.

Features distinguishing $\mathcal{G}^1(\Omega)$ from $\mathcal{G}^e(\Omega)$:

(1.0) Smooth dependence of R on (φ, x) rather than arbitrary dependence on φ and smoothness only with respect to x.

(1.1) Dependence of test objects on ε.

(1.2) Asymptotically vanishing moments of test objects as compared to the stronger condition $\phi(\varepsilon) \in \mathcal{A}_q(\mathbb{R}^s)$ for all ε (which would be the naïve analog of $\varphi \in \mathcal{A}_q(\mathbb{R}^s)$ in the case of $\mathcal{G}^e(\Omega)$).

Features distinguishing $\mathcal{G}^d(\Omega)$ from $\mathcal{G}^1(\Omega)$:

(2.1) Dependence of test objects also on $x \in \Omega$ (in fact, smooth dependence).

(2.2) In testing for moderateness, test objects for $\mathcal{G}^d(\Omega)$ can take arbitrary values in $\mathcal{A}_0(\mathbb{R}^s)$, independently of any moment condition.

Let us analyze briefly how compelling the above changes in the definitions in fact are if a diffeomorphism invariant algebra is to be obtained. We refrain from questioning (1.0), i.e., smoothness of R with respect to (φ, x), as well as from questioning the smoothness of test objects with respect to x in the sense of (2.1). Both properties being used in the proof of diffeomorphism invariance in an essential way, they are absolutely necessary from a pragmatic point ot of view to guarantee the smoothness of $(\hat{\mu}R)(S_\varepsilon\tilde{\phi}(\varepsilon, \tilde{x}), \tilde{x}) = R(S_\varepsilon\phi(\varepsilon, \mu\tilde{x}), \mu\tilde{x})$ with respect to $\tilde{x}$ (see section 7), to be sure. Of course, this does not amount to say that we have a formal proof that for the diffeomorphism invariance of an algebra, smoothness of R with respect to φ or of test objects with respect to x are logically necessary.

Smoothness of test objects with respect to ε definitly is not an issue of striking importance: The equivalence of conditions (B) and (C) in Theorem 10.5 (resp. of conditions (B′) and (C′) in Theorem 10.6) shows that test objects of the form $\phi \in \mathcal{C}_b^{[\infty,\Omega]}(I \times \Omega, \mathcal{A}_0(\mathbb{R}^s))$ give rise to $\mathcal{G}^d(\Omega)$ (via using them for testing moderateness resp. negligibility) independently of the assumption of smoothness of $\varepsilon \mapsto \phi(\varepsilon, x)$. With the appropriate respective modifications of the proof, this statement is valid for all types of test objects being dependent on ε or (ε, x), that is, it is true for all nine types $[z, Y]$ where z is one of εx or ε (see below for the definition of these types).

Next, if for a given diffeomorphism $\mu : \tilde{\Omega} \to \Omega$ the induced map $\hat{\mu} : \mathcal{E}(\Omega) \to \mathcal{E}(\tilde{\Omega})$ is to extend the usual action μ^* induced by μ on distributions then we necessarily have

to set $\hat\mu = \bar\mu^*$, i.e., $(\hat\mu R)(\tilde\varphi,\tilde x) = R(\bar\mu(\tilde\varphi,\tilde x))$ with $\bar\mu$ as defined in 5.7. For purposes of testing, we have, in turn, no other choice than to consider $\hat\mu_\varepsilon = \bar\mu_\varepsilon^*$ where

$$\bar\mu_\varepsilon(\tilde\varphi,\tilde x) = \left(\tilde\varphi\left(\frac{\mu^{-1}(\varepsilon.+\mu\tilde x)-\tilde x}{\varepsilon}\right)\cdot|\det D\mu^{-1}(\varepsilon.+\mu\tilde x)|\,,\,\mu\tilde x\right).$$

From $(\hat\mu_\varepsilon R)(\tilde\varphi,\tilde x) = R(\bar\mu_\varepsilon(\tilde\varphi,\tilde x))$ it is now evident that a moderate (resp. negligible) function R from the basic space has to accept test objects which are dependent on ε as well as on x ((1.1.) and (2.1)) if $\hat\mu R$ is still to be moderate (resp. negligible) (see the discussion preceding Theorem 7.14). (1.2) is compelling since the property that certain moments of a test object have to vanish simply is not invariant under $\bar\mu$ resp. $\bar\mu_\varepsilon$. The moments of the transformed test objects only vanish asymptotically. This has the further consequence that accepting (2.1) raises the question of how to handle asymptotically vanishing moments with respect to uniformity in $x\in\Omega$: Since all the definitions and theorems involve uniformity on compact subsets of Ω it seems reasonable to adopt this condition also for the asymptotically vanishing moment property, possibly even for all derivatives $\partial_x^\alpha\phi$ of a test object $\phi(\varepsilon,x)$. We will discuss several variants below.

So there only remains change (2.2) for which there seems to be no apparent necessity. To be sure, accepting (2.2) widens the range of permissible test objects, thereby in turn reducing $\mathcal{E}_M(\Omega)$ and $\mathcal{N}(\Omega)$ in size (see example 17.11 (i) below). Yet it has to be admitted that by this reduction, no generalized functions which are of interest either in the development of the theory or in applications are lost. Quite to the contrary, accepting (2.2) has the advantage that the definition of $\mathcal{E}_M(\Omega)$ becomes simpler and, above all, that considerable flexibility is gained in how to define $\mathcal{N}(\Omega)$, as respective glances at Theorems 7.9 and 13.1 reveal. Nevertheless, the preceding discussion leaves open the possibility that a diffeomorphism invariant Colombeau algebra $\mathcal{G}^2(\Omega)$ could be constructed avoiding (2.2). $\mathcal{G}^2(\Omega)$ would be closer to $\mathcal{G}^1(\Omega)$ than $\mathcal{G}^d(\Omega)$ is; the preceding considerations seem to suggest that passing from $\mathcal{G}^1(\Omega)$ to $\mathcal{G}^2(\Omega)$ would represent the minimal modification of $\mathcal{G}^1(\Omega)$ leading to a diffeomorphism invariant Colombeau algebra. In any case, a construction as envisaged above would yield a second example of a diffeomorphism invariant Colombeau algebra.

16.2 Classification of test objects

The term "test object" will always refer to some element of $\mathcal{C}_b^\infty(I\times\mathcal{A}_0(\mathbb{R}^s))$; apart from functions $\phi(\varepsilon,x)$, this formally also includes test objects of the form $\phi(\varepsilon)$ (depending only on ε) as well as elements φ of $\mathcal{A}_0(\mathbb{R}^s)$. From now on, we will write $\langle\xi^\alpha,\varphi(\xi)\rangle$ or even only $\langle\xi^\alpha,\varphi\rangle$ in place of $\int\xi^\alpha\varphi(\xi)\,d\xi$ for $\varphi\in\mathcal{D}(\mathbb{R}^s)$, $\alpha\in\mathbb{N}_0^s$.

16.1 Definition. *Let $q \in \mathbb{N}$. A function $\phi : I \to \mathcal{D}(\mathbb{R}^s)$ (possibly constant and/or depending also on other arguments, e.g., on $x \in \Omega$) is said to have* vanishing moments of order q *if $\langle \xi^\alpha, \phi(\varepsilon)(\xi)\rangle = 0$ for all $\alpha \in \mathbb{N}_0^s$ with $1 \le |\alpha| \le q$. It is said to have* asymptotically vanishing moments of order q *if $\langle \xi^\alpha, \phi(\varepsilon)(\xi)\rangle = O(\varepsilon^q)$ for all $\alpha \in \mathbb{N}_0^s$ with $1 \le |\alpha| \le q$. To which extent this estimate is assumed to hold uniformly with respect to, e.g., $x \in \Omega$ has to be specified separately (see below).*

A function ϕ taking values in $\mathcal{A}_0(\mathbb{R}^s)$ has vanishing moments of order q if and only if it takes values in $\mathcal{A}_q(\mathbb{R}^s)$, actually. To obtain a classification of Colombeau algebras lying in the range between $\mathcal{G}^d(\Omega)$ and (the smooth version of) $\mathcal{G}^e(\Omega)$ we introduce the symbols defined in the following list. They are meant to refer to test objects or to respective notions of moderateness and negligibility (based on test objects of the corresponding type) or, finally, to Colombeau algebras defined as quotients of the respective spaces of moderate functions.

Parametrization of test objects:

- [c] test objects being single elements ("constants") of $\mathcal{A}_0(\mathbb{R}^s)$ resp. $\mathcal{A}_q(\mathbb{R}^s)$
- [ε] test objects depending only on $\varepsilon \in I$
- [εx] test objects depending on $\varepsilon \in I$ as well as on $x \in \Omega$

Moments of test objects:

- [0] test objects taking values in $\mathcal{A}_0(\mathbb{R}^s)$ without any restriction on moments
- [A] test objects having asymptotically vanishing moments (this symbol always has to refer to test objects of type [ε])
- [V] test objects having vanishing moments, i.e., taking values in some $\mathcal{A}_q(\mathbb{R}^s)$

The following symbols make sense only for test objects of type [εx]; each of them indicates asymptotically vanishing moments of test objects, possibly also of their derivatives $\partial_x^\alpha \phi(\varepsilon, x)$, with the following respective specifications:

- [$\mathrm{A_l}$] uniformly on the particular $K \subset\subset \Omega$ on which R is being tested ("locally")
- [$\mathrm{A_g}$] uniformly on each $L \subset\subset \Omega$ ("globally")
- [$\mathrm{A_l^\infty}$] all derivatives uniformly on the particular $K \subset\subset \Omega$ on which R is being tested
- [$\mathrm{A_g^\infty}$] all derivatives uniformly on each $L \subset\subset \Omega$

If the compact set K on which R is being tested and/or the order q of the (asymptotic) vanishing of moments is to be specified, K resp. q will be put as subscript(s)

to the corresponding A-symbol, e.g., $[\mathrm{A_l}]_{K,q}$. Parametrization symbols may be combined with (suitable) moment symbols. If in a composed symbol $[z, Y]$ Y is one of the A-symbols then $z = \varepsilon$ resp. $z = \varepsilon x$, being redundant, will be omitted frequently.

Obviously, $[\mathrm{A_g^\infty}]_q$ implies $[\mathrm{A_l^\infty}]_{K,q}$ (for any $K \subset\subset \Omega$) and $[\mathrm{A_g}]_q$; each of the latter, in turn, implies $[\mathrm{A_l}]_{K,q}$. As the examples below show, none of the reverse implications is true.

16.2 Examples.

(i) Let $\Omega := \mathbb{R}$ and $K := [-1, +1]$. Define $\phi_1(\varepsilon, x) := \varphi + \varepsilon^q \sin(x|\ln \varepsilon|)\psi$ where $\varphi \in \mathcal{A}_q(\mathbb{R})$ and $\psi \in \mathcal{A}_{00}(\mathbb{R})$ with $\langle \xi^k, \psi(\xi)\rangle = \delta_{kq}$ for $k = 1, \ldots, q$. $\langle \xi^q, \partial_x \phi_1(\varepsilon, 0)(\xi)\rangle = \varepsilon^q |\ln \varepsilon|$ is not of order ε^q, yet every $\partial_x^n \phi_1(\varepsilon, x)$ has bounded image. Hence ϕ_1 is of type $[\mathrm{A_g}]_q$ and of type $[\mathrm{A_l}]_{K,q}$, yet neither of type $[\mathrm{A_g^\infty}]_q$ nor of type $[\mathrm{A_l^\infty}]_{K,q}$.

(ii) Let $K \subset\subset \Omega$ and set $\phi_2(\varepsilon, x) := \lambda(x)\varphi_1 + (1 - \lambda(x)\varphi_2)$ where $\varphi_1 \in \mathcal{A}_q(\mathbb{R}^s)$, $\varphi_2 \in \mathcal{A}_0(\mathbb{R}^s) \setminus \mathcal{A}_1(\mathbb{R}^s)$ and $\lambda \in \mathcal{D}(\Omega)$ such that $0 \le \lambda \le 1$ and $\lambda \equiv 1$ on an open neighborhood of K. Then for $q \in \mathbb{N}$, ϕ_2 is both of types $[\mathrm{A_l}]_{K,q}$ and $[\mathrm{A_l^\infty}]_{K,q}$, yet neither of type $[\mathrm{A_g}]_q$ nor of type $[\mathrm{A_g^\infty}]_q$.

16.3 Classification of full smooth Colombeau algebras

In the following, we will use the symbols introduced above to classify smooth Colombeau algebras with respect to the type of test objects used for testing moderateness. From a combinatorial point of view, there are eleven ways of performing this test, each corresponding to one of the eleven types of test objects. The following diagram displays these variants and the relations between them. The arrows are to be read as implications between the corresponding notions of moderateness or as inclusion relations between the corresponding sets of moderate functions (and similarly, for negligibility, as far as types [A] and [V] are concerned). They are *not* representing implications between the properties of test objects being of the particular types; a diagram of the latter kind would have to have the arrows reversed, of course.

$$
\begin{tikzcd}
[\varepsilon x,0] \arrow[rr] \arrow[d] & & [\varepsilon,0] \arrow[r] \arrow[dd] & [\mathrm{c},0] \arrow[ddd] \\
[\varepsilon x,\mathrm{A}_{\mathrm{l}}] \arrow[r] \arrow[d] & [\varepsilon x,\mathrm{A}_{\mathrm{g}}] \arrow[d] & & \\
[\varepsilon x,\mathrm{A}_{\mathrm{l}}^{\infty}] \arrow[r] & [\varepsilon x,\mathrm{A}_{\mathrm{g}}^{\infty}] \arrow[r] \arrow[d] & [\varepsilon,\mathrm{A}] \arrow[d] & \\
& [\varepsilon x,\mathrm{V}] \arrow[r] & [\varepsilon,\mathrm{V}] \arrow[r] & [\mathrm{c},\mathrm{V}]
\end{tikzcd}
$$

From the diagram, a useful extension of the characterizations of $R \in \mathcal{N}^d(\Omega)$ obtained so far[2] is immediate: Test objects in condition (4^∞) are of type $[\mathrm{A}_{\mathrm{l}}^{\infty}]_K$ where K is the compact set on which R is tested. This dependence on K of the class of admissible test objects (going back to [28], Theorem 18) might seem undesirable since this class rather ought to be defined universally for tests on arbitrary compact sets. However, it is clear from the diagram that our list of equivalent conditions could be extended by adding a further condition (5^∞), obtained from (4^∞) by replacing $[\mathrm{A}_{\mathrm{l}}^{\infty}]$ with $[\mathrm{A}_{\mathrm{g}}^{\infty}]$: It suffices to observe that (3°) and (4^∞) are based on test objects of types $[\varepsilon x, \mathrm{V}]$ and $[\varepsilon x, \mathrm{A}_{\mathrm{l}}^{\infty}]$, respectively. It will be a consequence of Corollary 16.8 below that a further extension by an analogous equivalent condition (6°), referring to type $[\mathrm{A}_{\mathrm{g}}]$, can be achieved.

One more glance at the diagram allows to clearly identify the obstacle against the diffeomorphism invariance of the algebra $\mathcal{G}^1(\Omega)$ defined in [13]: The Lemma in section 3 of that article only shows the μ-transform of test objects of type $[\varepsilon, \mathrm{A}]$ (being used in defining $\mathcal{G}^1(\Omega)$) to be of type $[\varepsilon x, \mathrm{A}_{\mathrm{g}}]$; yet, this is not sufficient for a positive outcome of the μ-transform of R being tested for, say, moderateness, provided R is assumed to be moderate. So it is Theorem **(T6)** of the blueprint outlined in section 3 which fails for $\mathcal{G}^1(\Omega)$.

Now, if $[X]$ and $[Y]$ are chosen from the set of the eleven types such that $[Y]$ is located "south to east" with respect to $[X]$ in the diagram above (i.e., if $\mathcal{E}_M[X] \subseteq \mathcal{E}_M[Y]$) and if, in addition, $[Y]$ is one of the types $[\mathrm{A}]$ or $[\mathrm{V}]$ then it easily checked that $\mathcal{E}_M[X]$ is an algebra (**(T2)**) containing $\mathcal{N}[Y] \cap \mathcal{E}_M[X]$ as an ideal (**(T3)**). Consequently, $\mathcal{E}_M[X]/(\mathcal{N}[Y] \cap \mathcal{E}_M[X])$ is an algebra. We shall refer to algebras arising in this way by the term "Colombeau-type algebras". Altogether there are 46 admissible choices of pairs $[X], [Y]$. In the following definition, we will specify eleven algebras

[2] (3°), (4^∞) in 7.9; (0°)–(2°) in 13.1; (A′)–(Z′) in 10.6; (C″), (Z″) in 10.7 (in each case assuming $R \in \mathcal{E}_M^d(\Omega)$, in addition).

of this kind, one for each type of moderateness. These will be the only ones we are to deal with in the sequel. They might be called "primary" since each of the remaining Colombeau-type algebras can be obtained as some subalgebra or some quotient algebra of one of them. Note, however, that the collection of these eleven algebras is not minimal in this respect (see Theorem 7.10).

16.3 Definition. *If* $[X]$ *is one of the types* [V] *or* [A] *define*

$$\mathcal{G}[X] := \mathcal{E}_M[X]/\mathcal{N}[X];$$

for types [0] *define*

$$\begin{aligned}
\mathcal{G}[\varepsilon x, 0] &:= \mathcal{E}_M[\varepsilon x, 0]/\big(\mathcal{N}[\varepsilon x, \mathrm{A}_\mathrm{l}^\infty] \cap \mathcal{E}_M[\varepsilon x, 0]\big), \\
\mathcal{G}[\varepsilon, 0] &:= \mathcal{E}_M[\varepsilon, 0]/\big(\mathcal{N}[\varepsilon, \mathrm{A}] \cap \mathcal{E}_M[\varepsilon, 0]\big), \\
\mathcal{G}[\mathrm{c}, 0] &:= \mathcal{E}_M[\mathrm{c}, 0]/\big(\mathcal{N}[\mathrm{c}, \mathrm{V}] \cap \mathcal{E}_M[\mathrm{c}, 0]\big).
\end{aligned}$$

(The open set Ω *has been omitted from the notation of the respective algebras.)*

We will refer to $\mathcal{G}[X]$ also by "the algebra of type $[X]$". Each of the algebras mentioned at the beginning of this section is one of the eleven algebras just defined: Denoting by $\mathcal{G}_0^e(\Omega)$ the "smooth part" of $\mathcal{G}^e(\Omega)$, i.e., the subalgebra formed by all members having a smooth representative $R \in \mathcal{C}^\infty(U^e(\Omega))$, it is easy to see that $\mathcal{G}_0^e(\Omega) = \mathcal{G}[\mathrm{c}, \mathrm{V}]$. $\mathcal{G}^1(\Omega)$ obviously is equal to $\mathcal{G}[\varepsilon, \mathrm{A}]$; the algebra $\mathcal{G}^2(\Omega)$ to be introduced in the following section is obtained as $\mathcal{G}[\varepsilon x, \mathrm{A}_\mathrm{g}^\infty]$. $\mathcal{G}^d(\Omega)$, finally, is given as $\mathcal{G}[\varepsilon x, 0]$. Observe that according to Theorem 7.9, $\mathcal{N}[\varepsilon x, \mathrm{A}_\mathrm{l}^\infty]$ can be replaced by $\mathcal{N}[\varepsilon x, \mathrm{V}]$ in the definition of $\mathcal{G}[\varepsilon x, 0]$. Moreover, it should be clear from Examples 7.7 and the discussion preceding them why $\mathcal{G}[\varepsilon x, 0]$ has *not* been defined as the quotient with respect to $\mathcal{N}[\varepsilon x, \mathrm{A}_\mathrm{l}] \cap \mathcal{E}_M[\varepsilon x, 0]$: This choice (corresponding to using condition (4°) of [28], Theorem 18) would invalidate part (iii) of **(T1)** and thus prevent ι from preserving the product of smooth functions.

Corollary 16.8 below will show that test objects of types $[\mathrm{A}_\mathrm{g}]$ and $[\mathrm{A}_\mathrm{g}^\infty]$, respectively, give rise to the same moderate resp. negligible functions. Moreover, it will follow from Theorem 17.4 that also test objects of type $[\mathrm{A}_\mathrm{l}^\infty]$ lead to the same respective notions of moderateness and negligibility as test objects of type $[\mathrm{A}_\mathrm{g}^\infty]$ do. This actually leaves us with nine possibly different algebras.

The diagram formed by the canonical homomorphisms between the resulting nine algebras is not isomorphic to the previous diagram: On the one hand, as mentioned above, $[\mathrm{A}_\mathrm{g}]$, $[\mathrm{A}_\mathrm{l}^\infty]$ and $[\mathrm{A}_\mathrm{g}^\infty]$ have to be merged to represent $\mathcal{G}^2(\Omega) = \mathcal{G}[\varepsilon x, \mathrm{A}_\mathrm{g}^\infty]$. On the other hand, there is no canonical homomorphism from $\mathcal{G}^d(\Omega) = \mathcal{G}[\varepsilon x, 0]$ into $\mathcal{G}[\varepsilon x, \mathrm{A}_\mathrm{l}]$ since $\mathcal{N}[\varepsilon x, \mathrm{A}_\mathrm{l}] \cap \mathcal{E}_M[\varepsilon x, 0]$—not containing any of $R(\varphi, x) := \langle \xi^\beta, \varphi(\xi)\rangle$—is strictly smaller than $\mathcal{N}[\varepsilon x, \mathrm{A}_\mathrm{l}^\infty] \cap \mathcal{E}_M[\varepsilon x, 0]$. We do have canonical homomorphisms,

however, both from $\mathcal{G}^d(\Omega) = \mathcal{G}[\varepsilon x, 0]$ and from $\mathcal{G}[\varepsilon x, \mathrm{A_l}]$ into $\mathcal{G}^2(\Omega) = \mathcal{G}[\varepsilon x, \mathrm{A}_\mathrm{g}^\infty]$. So we finally arrive at

$$\begin{array}{ccccccc} & & \mathcal{G}[\varepsilon x, 0] & \rightarrow & \mathcal{G}[\varepsilon, 0] & \rightarrow & \mathcal{G}[\mathrm{c}, 0] \\ & & \downarrow & & \downarrow & & \\ \mathcal{G}[\varepsilon x, \mathrm{A_l}] & \rightarrow & \mathcal{G}[\varepsilon x, \mathrm{A}_\mathrm{g}^\infty] & \rightarrow & \mathcal{G}[\varepsilon, \mathrm{A}] & & \downarrow \\ & & \downarrow & & \downarrow & & \\ & & \mathcal{G}[\varepsilon x, \mathrm{V}] & \rightarrow & \mathcal{G}[\varepsilon, \mathrm{V}] & \rightarrow & \mathcal{G}[\mathrm{c}, \mathrm{V}] \end{array}$$

When establishing **(T1)**–**(T8)** for $\mathcal{G}^2(\Omega)$ in the following section we will survey briefly which of these theorems is true for each of the (seven) algebras apart from $\mathcal{G}^2(\Omega) = \mathcal{G}[\varepsilon x, \mathrm{A}_\mathrm{g}^\infty]$ and $\mathcal{G}^d(\Omega) = \mathcal{G}[\varepsilon x, 0]$. Let us anticipate at this point the facts concerning **(T7)** and **(T8)**, i.e., diffeomorphism invariance: The following counterexamples of moderate functions R for which $\hat{\mu}R$ fails to be moderate for some diffeomorphism μ definitely eliminate six of the nine algebras from the class of possibly diffeomorphism invariant ones, beyond any pragmatic reasoning regarding techniques of proof.

16.4 Examples. Let $\Omega := \mathbb{R}$.

(i) The example $R_0(\varphi, x) := \exp(i \exp(\langle\varphi|\varphi\rangle))$ presented in [28] shows that all three algebras of type $[\varepsilon, Y]$ (Y=0,A,V), as well as the one of type $[\mathrm{c}, 0]$ are *not* diffeomorphism invariant.

(ii) Define $R_1(\varphi, x) := \langle \xi, \varphi(\xi)\rangle \cdot \exp(\langle\varphi|\varphi\rangle)$. Since R_1 vanishes on $\mathcal{A}_1(\mathbb{R}) \times \mathbb{R}$, it is moderate with respect to any type $[z, \mathrm{V}]$. Under the action induced by the diffeomorphism $\mu(x) := x + e^x$ of $\mathbb{R}$ onto itself, R_1 is transformed to a function $\hat{\mu}R_1$ which is not moderate with respect to any type $[z, \mathrm{V}]$ since the values attained by

$$(\hat{\mu}R_1)(S_\varepsilon\varphi, x) = \exp\Big(\frac{1}{\varepsilon}\int \frac{|\varphi(\xi)|^2}{1 + e^x e^{\varepsilon\xi}}\, d\xi\Big) \cdot \Big[\varepsilon \int \xi\varphi(\xi)\, d\xi + e^x \int (e^{\varepsilon\xi} - 1)\varphi(\xi)\, d\xi\Big]$$

are not of any order ε^{-N} ($n \in \mathbb{N}$) even for simple test objects of the form $\varphi \in \mathcal{A}_N(\mathbb{R})$. Therefore, $\hat{\mu}R_1$ does not pass the test for moderateness. This example excludes all types $[z, \mathrm{V}]$ from the class of diffeomorphism invariant algebras.

The details are left to the reader.

Thus we are left with the algebras of types $[\varepsilon x, 0]$, $[\varepsilon x, \mathrm{A}_{\mathrm{g}}^\infty]$ (together with the two equivalent types mentioned above) and $[\varepsilon x, \mathrm{A}_{\mathrm{l}}]$ as possible candidates for being diffeomorphism invariant. $[\varepsilon x, 0]$ giving rise to the algebra $\mathcal{G}^d(\Omega)$ introduced in section 7, we will define $\mathcal{G}^2(\Omega)$ in the following section on the basis of type $[\varepsilon x, \mathrm{A}_{\mathrm{g}}^\infty]$ and prove it to be a diffeomorphism invariant Colombeau algebra by establishing the corresponding Theorems **(T1)**–**(T8)**.

For a discussion of $\mathcal{G}[\mathrm{A}_{\mathrm{l}}]$, finally, we refer to the following section. It is clear from Example 7.7 that this algebra cannot be counted among the class of Colombeau algebras due to its multiplication not reproducing the product of smooth functions; moreover, we have to leave it open if it is a differential algebra at all since we do not know if $\mathcal{N}[\mathrm{A}_{\mathrm{l}}]$ is invariant under differentiation. Nevertheless, the spaces of moderate resp. negligible functions obtained from type $[\varepsilon x, \mathrm{A}_{\mathrm{l}}]$ test objects turn out to be diffeomorphism invariant. Despite the obvious faults of $\mathcal{G}[\mathrm{A}_{\mathrm{l}}]$, we have included type $[\mathrm{A}_{\mathrm{l}}]$ in our scheme, mainly to allow for a thorough discussion of condition (4°) of [28], Theorem 18.

Summarizing, the results of this and the following section show that $\mathcal{G}^d(\Omega)$ and $\mathcal{G}^2(\Omega)$ are the only diffeomorphism invariant Colombeau algebras among the eleven (resp. nine) algebras defined in 16.3.

To complete this section, it remains to prove that the tests based on types $[\mathrm{A}_{\mathrm{g}}^\infty]$ and $[\mathrm{A}_{\mathrm{g}}]$ are in fact equivalent. As demonstrated by Example 16.2 (i), there are test objects of type $[\mathrm{A}_{\mathrm{g}}]_q$ failing to be of type $[\mathrm{A}_{\mathrm{g}}^\infty]_q$. Nevertheless, both these classes of test objects do give rise to the same moderate resp. negligible functions. This fact will emerge as an immediate corollary from the following theorem.

16.5 Theorem. *Let $\phi \in \mathcal{C}_b^\infty(I \times \Omega, \mathcal{A}_0(\mathbb{R}^s))$ and let $2 \le q \in \mathbb{N}$. If ϕ is of type $[\mathrm{A}_{\mathrm{g}}]_q$ then it also is of type $[\mathrm{A}_{\mathrm{g}}^\infty]_{q-1}$.*

For the proof we need two lemmas.

16.6 Lemma. *Let $c : I \times \Omega \to \mathbb{R}$ have second partial derivatives $\partial_i^2 c$ for some $i \in \{1, \dots, s\}$ ($\partial_i = \frac{\partial}{\partial x_i}$). Let $q > 0$, $0 \le r < 1$ and assume that $K \subset\subset L \subset\subset \Omega$. If $\sup_{x \in L} |c(\varepsilon, x)| = O(\varepsilon^q)$ and $\sup_{x \in L} |\partial_i^2 c(\varepsilon, x)| = O(\varepsilon^{rq})$ then $\sup_{x \in K} |\partial_i c(\varepsilon, x)| = O(\varepsilon^{\frac{1+r}{2}q})$.*

Proof. We consider values of $\varepsilon \in I$ which are less than $\mathrm{dist}(K, \partial L)$; set $p := q\frac{1-r}{2}$. For $x \in K$, $x + \varepsilon^p e_i \in L$. Taylor's Theorem yields

$$c(\varepsilon, x + \varepsilon^p e_i) \;=\; c(\varepsilon, x) + \varepsilon^p \partial_i c(\varepsilon, x) + \varepsilon^{2p} \frac{1}{2} \partial_i^2 c(\varepsilon, x_\theta)$$

where $x_\theta = x + \theta\varepsilon^p e_i$ for some $\theta \in (0,1)$; note that also $x_\theta \in L$. Consequently,

$$\partial_i c(\varepsilon, x) = \varepsilon^{-p} \underbrace{(c(\varepsilon, x+\varepsilon^p e_i) - c(\varepsilon, x))}_{O(\varepsilon^q)} - \varepsilon^p \underbrace{\frac{1}{2}\partial_i^2 c(\varepsilon, x_\theta)}_{O(\varepsilon^{rq})} = O(\varepsilon^{\frac{1+r}{2}q}),$$

uniformly for $x \in K$. □

For the second lemma, we inductively define a sequence of numbers r_k by setting $r_1 := 0$, $r_{k+1} := \frac{(1+r_k)^2}{4}$ ($k \in \mathbb{N}$). Being strictly increasing and bounded by 1, this sequence is convergent, its limit being equal to 1.

16.7 Lemma. *For every $k \in \mathbb{N}$ the following holds: Let $c : I \times \Omega \to \mathbb{R}$ be smooth with respect to the variable x_i ($x = (x_1, \ldots, x_s) \in \Omega$) for some $i \in \{1, \ldots, s\}$. Let $q > 0$ and $K \subset\subset L \subset\subset \Omega$. If $\sup_{x\in L} |c(\varepsilon, x)| = O(\varepsilon^q)$ and $\sup_{x\in L} |\partial_i^m c(\varepsilon, x)| = O(1)$ for all $m \in \mathbb{N}$ then $\sup_{x\in K} |\partial_i c(\varepsilon, x)| = O(\varepsilon^{\frac{1+r_k}{2}q})$.*

Proof. Proceeding by induction, the case $k = 1$ is immediate from Lemma 16.6 by setting $r := r_1 = 0$. Assume the statement of the lemma to be true for a particular $k \in \mathbb{N}$. Let c, i, q, K, L be as specified. Choose K_1, K_2 as to satisfy $K \subset\subset K_1 \subset\subset K_2 \subset\subset L$. From $\sup_{x\in L} |c(\varepsilon, x)| = O(\varepsilon^q)$ and $\sup_{x\in L} |\partial_i^m c(\varepsilon, x)| = O(1)$ for all $m \in \mathbb{N}$ we deduce, by assumption, $\sup_{x\in K_2} |\partial_i c(\varepsilon, x)| = O(\varepsilon^{\frac{1+r_k}{2}q})$. Applying the statement of the lemma (for the particular value of k under consideration) once more, this time to the function $\partial_i c$, with $\frac{1+r_k}{2}q$ in place of q and for the pair K_1, K_2 of compact sets, we obtain $\sup_{x\in K_1} |\partial_i^2 c(\varepsilon, x)| = O(\varepsilon^{(\frac{1+r_k}{2})^2 q})$. In a last step, we apply Lemma 16.6 to conclude that $\sup_{x\in K} |\partial_i c(\varepsilon, x)| = O(\varepsilon^{\bar{r}q})$ where $\bar{r} = \frac{1}{2}(1 + \frac{(1+r_k)^2}{4}) = \frac{1+r_{k+1}}{2}$, thereby showing the statement of the lemma to be true also for $k+1$. □

Proof of Theorem 16.5. Let $\phi \in \mathcal{C}_b^\infty(I \times \Omega, \mathcal{A}_0(\mathbb{R}^s))$ be of type $[\mathrm{A_g}]_q$ where $2 \le q \in \mathbb{N}$. Denoting $\langle \xi^\alpha, \phi(\varepsilon, x)(\xi)\rangle$ by $c_\alpha(\varepsilon, x)$ ($\alpha \in \mathbb{N}_0^s$), we have to show that

$$\sup_{x\in K} |\langle \xi^\alpha, \partial_x^\beta \phi(\varepsilon, x)(\xi)\rangle| = \sup_{x\in K} |\partial^\beta c_\alpha(\varepsilon, x)| = O(\varepsilon^{q-1})$$

for $1 \le |\alpha| \le q-1$ and all $K \subset\subset \Omega$, $\beta \in \mathbb{N}_0^s$. Fix $\alpha \in \mathbb{N}_0^s$ satisfying $1 \le |\alpha| \le q$. By assumption, we have $\sup_{x\in L} |c_\alpha(\varepsilon, x)| = O(\varepsilon^q)$ and $\sup_{x\in L} |\partial^\beta c_\alpha(\varepsilon, x)| = O(1)$ for all $L \subset\subset \Omega$ and all $\beta \in \mathbb{N}_0^s$. Since $q\frac{1+r_k}{2} \to q$ as $k \to \infty$, Lemma 16.7 yields that $\sup_{x\in K} |\partial_i c_\alpha(\varepsilon, x)| = O(\varepsilon^{q-\frac{1}{2}})$ for every $K \subset\subset \Omega$ and any $i = 1, \ldots, s$. Noting that also

$(q-\frac{1}{2})\frac{1+r_k}{2} \to (q-\frac{1}{2})$, the same argument, applied to $\partial_i c_\alpha$ and ∂_j ($j = 1, \ldots, s$) in place of c_α and ∂_i, respectively, shows that $\sup_{x\in K} |\partial_j \partial_i c_\alpha(\varepsilon, x)| = O(\varepsilon^{q-(\frac{1}{2}+\frac{1}{4})})$, again for every $K \subset\subset \Omega$ and any $i, j = 1, \ldots, s$. By induction, we obtain $\sup_{x\in K} |\partial^\beta c_\alpha(\varepsilon, x)| = O(\varepsilon^{q-q_\beta})$ for all $\beta \in \mathbb{N}_0^s$ where $q_\beta = \sum_{i=1}^{|\beta|} 2^{-i} < 1$. From this we finally conclude that $\sup_{x\in K} |\partial^\beta c_\alpha(\varepsilon, x)| = O(\varepsilon^{q-1})$ for all $\beta \in \mathbb{N}_0^s$ and all $K \subset\subset \Omega$. □

16.8 Corollary. *Let $R \in \mathcal{E}(\Omega)$. R is moderate (resp. negligible) with respect to type $[\mathrm{A_g}]$ if and only if it is moderate (resp. negligible) with respect to type $[\mathrm{A_g^\infty}]$.*

Proof. Necessity of the condition being obvious, let us show sufficiency. Assuming R to be moderate with respect to type $[\mathrm{A_g^\infty}]$, fix $\alpha \in \mathbb{N}_0^s$, $K \subset\subset \Omega$. Choose $N_1 \in \mathbb{N}$ such that $\partial^\alpha(R(S_\varepsilon\phi_1(\varepsilon, x), x)) = O(\varepsilon^{-N_1})$ holds for every test object ϕ_1 of type $[\mathrm{A_g^\infty}]_{N_1}$, uniformly on K. Now set $N := N_1 + 1$ and pick a test object ϕ of type $[\mathrm{A_g}]_N$. By Theorem 16.5, ϕ is of type $[\mathrm{A_g^\infty}]_{N-1}$, i.e., of type $[\mathrm{A_g^\infty}]_{N_1}$. Due to our choice of N_1, $\partial^\alpha(R(S_\varepsilon\phi(\varepsilon, x), x)) = O(\varepsilon^{-N_1})$ resp. $O(\varepsilon^{-N})$ follow. A similar argument applies to negligibility of R. □

17 The algebra $\mathcal{G}^2$; classification results

The algebra $\mathcal{G}^2(\Omega)$ of type $[\varepsilon x, \mathrm{A_g^\infty}]$ to be analyzed below results from the algebra $\mathcal{G}^1(\Omega) = \mathcal{G}[\varepsilon, \mathrm{A}]$ of [13] by applying the minimal modification necessary to obtain diffeomorphism invariance. Recall that a test object $\phi \in \mathcal{C}_b^\infty(I \times \Omega, \mathcal{A}_0(\mathbb{R}^s))$ is said to be of type $[\varepsilon x, \mathrm{A_g^\infty}]_q$ if $\sup_{x\in K} |\langle \xi^\alpha, \partial_x^\beta \phi(\varepsilon, x)(\xi)\rangle| = O(\varepsilon^q)$ for every $K \subset\subset \Omega$, $\beta \in \mathbb{N}_0^s$ and $\alpha \in \mathbb{N}_0^s$ with $1 \le |\alpha| \le q$. Moderateness resp. negligibility of $R \in \mathcal{E}(\Omega) = \mathcal{C}^\infty(U(\Omega))$ are defined as follows (where $K \subset\subset \Omega$ and $\alpha \in \mathbb{N}_0^s$):

17.1 Definition. *$R \in \mathcal{E}(\Omega)$ is moderate with respect to type $[\varepsilon x, \mathrm{A_g^\infty}]$ if the following condition is satisfied:*

$$\forall K\ \forall \alpha\ \exists N \in \mathbb{N}\ \forall \phi \in \mathcal{C}_b^\infty(I \times \Omega, \mathcal{A}_0(\mathbb{R}^s)) \text{ which are of type } [\varepsilon x, \mathrm{A_g^\infty}]_N:$$

$$\sup_{x\in K} |\partial^\alpha(R(S_\varepsilon\phi(\varepsilon, x), x))| = O(\varepsilon^{-N}).$$

17.2 Definition. *$R \in \mathcal{E}(\Omega)$ is negligible with respect to type $[\varepsilon x, \mathrm{A}_{\mathrm{g}}^{\infty}]$ if the following condition is satisfied:*

$$\forall K \ \forall \alpha \ \forall n \in \mathbb{N} \ \exists q \in \mathbb{N} \ \forall \phi \in \mathcal{C}_b^{\infty}(I \times \Omega, \mathcal{A}_0(\mathbb{R}^s)) \text{ which are of type } [\varepsilon x, \mathrm{A}_{\mathrm{g}}^{\infty}]_q :$$

$$\sup_{x \in K} |\partial^{\alpha}(R(S_{\varepsilon}\phi(\varepsilon, x), x))| = O(\varepsilon^n).$$

Since we are dealing with $\mathcal{G}^2(\Omega)$ exclusively in the following, we simply denote the sets of moderate resp. negligible functions in the sense of the preceding definitions by $\mathcal{E}_M(\Omega)$, $\mathcal{N}(\Omega)$. To establish $\mathcal{G}^2(\Omega)$ as a diffeomorphism invariant Colombeau algebra we have to convince ourselves that Theorems **(T1)**–**(T8)** of the scheme presented in section 3 are true on the basis of the preceding definitions (compare section 7 for the detailed elaboration of these theorems in the case of $\mathcal{G}^d(\Omega)$). Though our main interest will be focused on type $[\mathrm{A}_{\mathrm{g}}^{\infty}]$, of course, for each of **(T1)**–**(T8)** we will specify for which of the remaining types (apart from $[\varepsilon x, \mathrm{A}_{\mathrm{g}}^{\infty}]$ and $[\varepsilon x, 0]$) it holds as well.

To start with, (i) and (ii) of **(T1)** follow from the corresponding statements with respect to $\mathcal{G}^d(\Omega)$ (7.4 ,(i),(ii)) for all types since $[\varepsilon x, 0]$ generates the smallest one of all spaces $\mathcal{E}_M[X]$. We already know from Example 7.7 that (iii) of **(T1)** is not satisfied for type $[\mathrm{A}_{\mathrm{l}}]$. For all the remaining types, however, the corresponding statement follows immediately from part (iii) of 7.4 by observing that $\mathcal{N}[\varepsilon x, \mathrm{V}] \cap \mathcal{E}_M[\varepsilon x, 0] = \mathcal{N}[\varepsilon x, \mathrm{A}_{\mathrm{l}}^{\infty}] \cap \mathcal{E}_M[\varepsilon x, 0] = \mathcal{N}[\varepsilon x, \mathrm{A}_{\mathrm{g}}^{\infty}] \cap \mathcal{E}_M[\varepsilon x, 0]$ is contained in each space $\mathcal{N}[Y]$ where $[Y]$ is different from $[\mathrm{A}_{\mathrm{l}}]$. (The preceding equalities are due to Theorem 7.9 resp. to $(4^{\infty}) \Leftrightarrow (5^{\infty})$ derived in the preceding section.) Finally, the proof of part (iv) of **(T1)** given in section 7 for $\mathcal{G}^d(\Omega)$ uses test objects of type $[\mathrm{c}, \mathrm{V}]$ (generating the largest one of all spaces $\mathcal{N}[X]$) and therefore is valid for all types.

Theorems **(T2)** and **(T3)** are immediate from Leibniz' rule for all types.

As it had been the case for $\mathcal{G}^d(\Omega)$, **(T4)**–**(T6)** are the hard ones to prove also for $\mathcal{G}^2(\Omega)$. Fortunately, **(T6)** can be taken from section 7 with only a slight modification, as we will see. For **(T4)** and **(T5)**, however, we need analogs of Theorems 7.12 and 7.13 for type $[\mathrm{A}_{\mathrm{g}}^{\infty}]$ allowing to express moderateness resp. neglibility of R in terms of differentials of R_{ε}. To this end, we have to introduce appropriate classes of sets corresponding to the bounded subsets $B \subseteq \mathcal{D}(\mathbb{R}^s)$ occurring in Theorems 7.12 and 7.13. For any closed affine subspace E_1 of a locally convex space E, let $\mathcal{C}_b^{\infty}(I, E_1)$ denote the set of all smooth maps $\varphi : I \to E_1$ having bounded image.

17.3 Definition. *Let $k \in \mathbb{N}_0$, $q \in \mathbb{N}$.*
A (k,q)-class is a subset $\mathcal{B}$ of $\mathcal{C}_b^{\infty}(I, \mathcal{A}_0(\mathbb{R}^s)) \times \left[\mathcal{C}_b^{\infty}(I, \mathcal{A}_{00}(\mathbb{R}^s))\right]^k$ satisfying the following conditions:

(i) The set $\{\psi_0(\varepsilon), \ldots, \psi_k(\varepsilon) \mid (\psi_0, \ldots, \psi_k) \in \mathcal{B},\ \varepsilon \in I\}$ is bounded in $\mathcal{D}(\mathbb{R}^s)$;

(ii) $\sup\limits_{(\psi_i)\in\mathcal{B}} \sup\limits_{i=0,\ldots,s} |\langle \xi^\beta, \psi_i(\varepsilon)(\xi)\rangle| = O(\varepsilon^q)$ for all $\beta \in \mathbb{N}_0^s$ with $1 \le |\beta| \le q$.

Note that $(\psi_0, \ldots, \psi_k) \in \mathcal{C}_b^\infty(I, \mathcal{A}_0(\mathbb{R}^s)) \times \left[\mathcal{C}_b^\infty(I, \mathcal{A}_{00}(\mathbb{R}^s))\right]^k$ forms a (k,q)-class $\{(\psi_0, \ldots, \psi_k)\}$ (consisting of a single element) if and only if each of $\psi_0, \ldots, \psi_k$ has asymptotically vanishing moments of order q. The following results are established by combining techniques of the respective proofs of Theorem 17 of [28] and of Theorem 10.5.

17.4 Theorem. *Let $R \in \mathcal{E}(\Omega)$. R is moderate of type $[\mathrm{A}_\mathrm{g}^\infty]$ if and only if the following condition is satisfied:*

$$\forall K \subset\subset \Omega\ \forall \alpha \in \mathbb{N}_0^d\ \forall k \in \mathbb{N}_0\ \exists N \in \mathbb{N} \text{ such that for each } (k,N)\text{-class } \mathcal{B}:$$

$$\sup_{(\psi_i)\in\mathcal{B}} \sup_{x\in K} |\partial^\alpha \mathrm{d}_1^k R_\varepsilon(\psi_0(\varepsilon), x)(\psi_1(\varepsilon), \ldots, \psi_k(\varepsilon))| = O(\varepsilon^{-N}).$$

Moreover, for given K, α, k, N the preceding condition is satisfied for all (k,N)-classes $\mathcal{B}$ if and only if it is satisfied for all (k,N)-classes consisting of a single element $(\psi_0, \ldots, \psi_k)$. Therefore, the uniformity requirement with respect to $\mathcal{B}$ can as well be omitted from the characterization of moderateness given above.

17.5 Theorem. *Let $R \in \mathcal{E}(\Omega)$. R is negligible of type $[\mathrm{A}_\mathrm{g}^\infty]$ if and only if the following condition is satisfied:*

$$\forall K \subset\subset \Omega\ \forall \alpha \in \mathbb{N}_0^d\ \forall k \in \mathbb{N}_0\ \forall n \in \mathbb{N}\ \exists q \in \mathbb{N} \text{ such that for each } (k,q)\text{-class } \mathcal{B}:$$

$$\sup_{(\psi_i)\in\mathcal{B}} \sup_{x\in K} |\partial^\alpha \mathrm{d}_1^k R_\varepsilon(\psi_0(\varepsilon), x)(\psi_1(\varepsilon), \ldots, \psi_k(\varepsilon))| = O(\varepsilon^{n}).$$

Moreover, for given K, α, k, n, q the preceding condition is satisfied for all (k,q)-classes $\mathcal{B}$ if and only if it is satisfied for all (k,q)-classes consisting of a single element $(\psi_0, \ldots, \psi_k)$. Therefore, the uniformity requirement with respect to $\mathcal{B}$ can as well be omitted from the characterization of negligibility given above.

The proofs of Theorems 17.4 and 17.5 are deferred to the end of this section.

17.6 Corollary. *Let $R \in \mathcal{E}(\Omega)$. R is moderate (resp. negligible) with respect to type $[\mathrm{A}_\mathrm{g}^\infty]$ if and only if it is moderate (resp. negligible) with respect to type $[\mathrm{A}_\mathrm{l}^\infty]$.*

Proof. Sufficiency of the condition being obvious, let us show necessity. Supposing R to be moderate with respect to type $[\mathrm{A}_\mathrm{g}^\infty]$, the differentials of R satisfy the condition of Theorem 17.4. For testing R on some $K \subset\subset \Omega$ as to moderateness with respect to type $[\mathrm{A}_\mathrm{l}^\infty]$, we have to consider test objects just of that type. Now it is exactly the easy part of the very proof of 17.4 which shows that this test gives a positive answer. The same argument applies to negligibility. $\square$

From the preceding Corollary and Corollary 16.8, we see that all three types $[\mathrm{A}_\mathrm{g}^\infty]$, $[\mathrm{A}_\mathrm{g}]$ and $[\mathrm{A}_\mathrm{l}^\infty]$ give rise to the same notions of moderateness resp. negligibility, hence to the same Colombeau algebras. This fact also constitutes one of the key ingredients for obtaining an intrinsic description of the algebra $\mathcal{G}^d(\Omega)$ on manifolds: The property of a test object living on the manifold to have asymptotically vanishing moments can be formulated in intrinsic terms, indeed ([26], Definition 3.5); yet it would be virtually unmanageable to deal with the latter property also for derivatives of this test object, which, of course, are to be understood in this general case as appropriate Lie derivatives with respect to smooth vector fields. Now Corollaries 16.8 and 17.6 allow to dispense with derivatives of test objects as regards the asymptotic vanishing of the moments, provided all $K \subset\subset \Omega$ are taken into account ([26], Corollary 4.5).

The next corollary might come as a bit of a surprise since we are already used to type $[\mathrm{A}_\mathrm{l}]$ displaying rather bad properties. Observe that it (necessarily, compare Example 7.7) only refers to moderateness. The case at hand seems to be the only one where a certain symmetry between $\mathcal{E}_M$ and $\mathcal{N}$ is broken.

17.7 Corollary. *Let $R \in \mathcal{E}(\Omega)$. R is moderate with respect to type $[\mathrm{A}_\mathrm{l}]$ if and only if it is moderate with respect to type $[\mathrm{A}_\mathrm{g}^\infty]$ (resp. $[\mathrm{A}_\mathrm{l}^\infty]$ resp. $[\mathrm{A}_\mathrm{g}]$).*

Proof. Necessity of the condition being obvious this time, let us show sufficiency. Suppose R to be moderate with respect to type $[\mathrm{A}_\mathrm{g}^\infty]$ and let $K \subset\subset \Omega$, $\alpha \in \mathbb{N}_0^s$ be given. According to Theorem 17.4, choose $N \in \mathbb{N}$ such that for every $k = 0, 1, \ldots, |\alpha|$, for every $\beta \in \mathbb{N}_0^s$ with $0 \leq |\beta| \leq |\alpha|$ and for every (k, N)-class $\mathcal{B}$,

$$\sup_{(\psi_i)\in\mathcal{B}} \sup_{x\in K} |\partial^\alpha \mathrm{d}_1^k R_\varepsilon(\psi_0(\varepsilon), x)(\psi_1(\varepsilon), \ldots, \psi_k(\varepsilon))| = O(\varepsilon^{-N}).$$

For any test object ϕ of type $[\mathrm{A_l}]_{K,N(1+|\alpha|)}$, it now follows

$$\begin{aligned}
&\sup_K |\partial^\alpha(R_\varepsilon(\phi(\varepsilon,x),x))| \\
&\qquad = \sup_K \Big|\sum_{\beta,m}(\partial^\beta \mathrm{d}_1^m R_\varepsilon)(\phi(\varepsilon,x),x)(\partial^{\gamma_1}\phi(\varepsilon,x),\dots,\partial^{\gamma_m}\phi(\varepsilon,x))\Big| \\
&\qquad = \sup_K \Big|\sum_{\beta,m}(\partial^\beta \mathrm{d}_1^m R_\varepsilon)(\phi(\varepsilon,x),x)(\varepsilon^N\partial^{\gamma_1}\phi(\varepsilon,x),\dots,\varepsilon^N\partial^{\gamma_m}\phi(\varepsilon,x))\cdot\varepsilon^{-mN}\Big| \\
&\qquad = O(\varepsilon^{-N-|\alpha|N})
\end{aligned}$$

since, for every $m \in \mathbb{N}_0$, the finite sequences $(\phi(\varepsilon,x),\varepsilon^N\partial^{\gamma_1}\phi(\varepsilon,x),\dots,\varepsilon^N\partial^{\gamma_m}\phi(\varepsilon,x))$ (with x ranging over K) form an (m,N)-class. □

Now the proofs of **(T4)** and **(T5)**, that is, of the invariance of $\mathcal{E}_M[\mathrm{A_g^\infty}]$ and $\mathcal{N}[\mathrm{A_g^\infty}]$ with respect to differentiation, follow from Theorems 17.4 and 17.5 in precisely the same way as they have been achieved in section 7 for the builiding blocks of $\mathcal{G}^d(\Omega)$ by means of Theorems 7.12 and 7.13. Digressing once more from the proof of $\mathcal{G}^2(\Omega)$ being a diffeomorphism invariant Colombeau algebra, let us deal with invariance under differentiation for the remaining types of algebras: Types $[\mathrm{A_g}]$ and $[\mathrm{A_l^\infty}]$ as well as the case of $\mathcal{E}_M[\mathrm{A_l}]$ are covered by Corollaries 16.8, 17.6 and 17.7, respectively. Moreover, it easy to check that **(T4)** and **(T5)** are true for all types $[\varepsilon]$ and [c]. An appropriate modification of Theorem 17 of [28] putting $\mathcal{A}_N(\mathbb{R}^s)$ resp. $\mathcal{A}_q(\mathbb{R}^s)$ in place of $\mathcal{A}_0(\mathbb{R}^s)$ and employing the techniques used in the proof of (A) $\Leftrightarrow$ (C) of Theorem 10.5 establishes the respective results to hold also for type $[\varepsilon x, V]$. Type $[\varepsilon x, 0]$ being covered by section 7, we are left only with $\mathcal{N}[\mathrm{A_l}]$ to be discussed. However, lacking an analog of Theorem 7.13 resp. of Theorem 17.5 for type $[\mathrm{A_l}]$ we are not in a position to express negligibility of R with respect to this type in terms of differentials of R_ε. This tool, however, was the basis for deducing invariance under differentiation. So for the time being, we find $\mathcal{N}[\mathrm{A_l}]$ to be the only one among all 11+8 (to be precise, 8+6 pairwise different) spaces $\mathcal{E}_M[X]$ resp. $\mathcal{N}[X]$ for which the invariance with respect to differentiation has to remain an open problem.

Finally, let us consider the question of diffeomorphism invariance. As an inspection of the structure of the proof of Theorem 7.14 of section 7 reveals, this theorem actually shows the μ-transform of test objects of types $[\mathrm{A_l}]$, $[\mathrm{A_g}]$, $[\mathrm{A_l^\infty}]$, $[\mathrm{A_g^\infty}]$ to be of the same type again, respectively, thereby establishing **(T6)** in all four cases. Moreover, we see that on the basis of Corollaries 16.8 and 17.6 even a weaker version of the last statement of Theorem 7.14, referring only to types $[\mathrm{A_l}]$ and $[\mathrm{A_g}]$, would suffice to obtain diffeomorphism invariance for all four types $[\varepsilon x, \mathrm{A}]$ (and, still, for $\mathcal{E}_M[\varepsilon x, 0]$; hence this would completely satisfy also the needs of section 7 dealing with $\mathcal{G}^d(\Omega)$!): Derivatives $\partial_x^\alpha\phi$ $(\alpha \neq 0)$ could be dispensed with in Theorem 7.14 and its proof.

Recall that our proofs of **(T7)** and **(T8)** in section 7 (stating the invariance of moderateness resp. negligibility under the action induced by a diffeomorphism) were based on the equivalence of conditions (C) and (Z) in Theorem 10.5 (resp. of (C$''$) and (Z$''$) in Corollary 10.7) which, in turn, used the extension of paths $\phi(\varepsilon, x)$ provided by Proposition 10.4. Now each of the four types $[\varepsilon x, \mathrm{A}]$ is preserved by the extension process $\phi \mapsto \tilde{\phi}$. Thus the respective analogs of (C)$\Leftrightarrow$(Z) can be shown to hold for all types $[\varepsilon x, \mathrm{A}]$ by the methods employed in 10.5.

Now the proofs of **(T7)** and **(T8)**, respectively, are literally the same for all four types $[\varepsilon x, \mathrm{A}]$ as for the algebra $\mathcal{G}^d(\Omega)$ treated in section 7. The proof of **(T8)** is even simpler in the present case since we do not have to bother with bridging the gap between vanishing moments (as used in Definition 7.3) and asymptotically vanishing moments (as occurring in Theorem 7.14) which has been accomplished in section 7 by means of the equivalence $(3^\circ)\Leftrightarrow(4^\infty)$ provided by Theorem 7.9.

Summarizing, we have established

17.8 Theorem. *$\mathcal{G}^2(\Omega)$ is a diffeomorphism invariant Colombau algebra which can be obtained by using test objects of any of the types $[\mathrm{A}_\mathrm{g}^\infty]$, $[\mathrm{A}_\mathrm{g}]$ or $[\mathrm{A}_\mathrm{l}^\infty]$.*

Test objects of type $[\mathrm{A}_\mathrm{l}]$, on the other hand, give rise to a diffeomorphism invariant algebra which does not preserve the product of smooth functions via ι and for which it remains open if it is a differential algebra at all. Moreover, we have shown that each of the remaining six algebras (apart from $\mathcal{G}^d(\Omega) = \mathcal{G}[\varepsilon x, 0]$) satisfies **(T1)**–**(T5)**, yet fails to be diffeomorphism invariant.

Also for the algebra $\mathcal{G}^2(\Omega)$ it is true that in characterizing the negligibility of $R \in \mathcal{E}_M(\Omega)$ in terms of the differentials of R_ε, derivatives can be dispensed with, those with respect to φ as well as those with respect to x. The numbering of the conditions in the following theorem corresponds to that of Theorem 13.1.

17.9 Theorem. *Let $R \in \mathcal{E}(\Omega)$ be moderate with respect to type $[\mathrm{A}_\mathrm{g}^\infty]$ (resp. with respect to types $[\mathrm{A}_\mathrm{g}]$, $[\mathrm{A}_\mathrm{l}^\infty]$). Then R is negligible with respect to any one of these types if and only if one of the following (equivalent) conditions is satisfied:*

(0_A°) $\forall K \subset\subset \Omega\ \forall n \in \mathbb{N}\ \exists q \in \mathbb{N}$ *such that for each* $(0,q)$*-class* $\mathcal{B}$*:*

$$\sup_{(\psi_0)\in\mathcal{B}} \sup_{x\in K} |R_\varepsilon(\psi_0(\varepsilon), x)| = O(\varepsilon^n).$$

(1_A°) $\forall K \subset\subset \Omega\ \forall \alpha \in \mathbb{N}_0^d\ \forall n \in \mathbb{N}\ \exists q \in \mathbb{N}$ *such that for each* $(0,q)$*-class* $\mathcal{B}$*:*

$$\sup_{(\psi_0)\in\mathcal{B}} \sup_{x\in K} |\partial^\alpha R_\varepsilon(\psi_0(\varepsilon), x)| = O(\varepsilon^n).$$

(2°_{A}) $\forall K \subset\subset \Omega\ \forall \alpha \in \mathbb{N}_0^d\ \forall k \in \mathbb{N}_0\ \forall n \in \mathbb{N}\ \exists q \in \mathbb{N}$ *such that for each (k,q)-class $\mathcal{B}$:*

$$\sup_{(\psi_i)\in\mathcal{B}} \sup_{x\in K} |\partial^\alpha \mathrm{d}_1^k R_\varepsilon(\psi_0(\varepsilon), x)(\psi_1(\varepsilon), \dots, \psi_k(\varepsilon))| = O(\varepsilon^n).$$

In each of the preceding conditions, the uniformity requirement with respect to $\mathcal{B}$ can as well be omitted without changing the content of the condition, regardless of the moderateness of R.

Proof. Due to Corollaries 16.8 and 17.6 it does not matter which of the three types is being considered. For $R \in \mathcal{E}_M[\mathrm{A}_\mathrm{g}^\infty]$, ($2^\circ_{\mathrm{A}}$) is equivalent to negligibility with respect to $[\mathrm{A}_\mathrm{g}^\infty]$ by Theorem 17.5. (2°_{A}) $\Rightarrow$ (1°_{A}) $\Rightarrow$ (0°_{A}) is trivial; (0°_{A}) $\Rightarrow$ (1°_{A}) and (1°_{A}) $\Rightarrow$ (2°_{A}) can be established by carefully replacing bounded subsets of $\mathcal{A}_0(\mathbb{R}^s)$ resp. of $\mathcal{A}_{00}(\mathbb{R}^s)$ by appropriately chosen (k,q)-classes in the respective proofs of Theorem 13.1 and part (1°) $\Rightarrow$ (2°) of Theorem 18 of [28]. As far as the proof of (1°_{A}) $\Rightarrow$ (2°_{A}) (proceeding by induction with respect to k) is concerned, the most delicate task in this respect consists in choosing appropriate $(k{+}1,q)$- resp. $(k{-}1,q)$-classes $\mathcal{B}_{+1}, \mathcal{B}_{-1}$ to be used in connection with $\partial^\alpha \mathrm{d}_1^{k+1} R_\varepsilon$ resp. $\partial^\alpha \mathrm{d}_1^{k-1} R_\varepsilon$ when $\partial^\alpha \mathrm{d}_1^k R_\varepsilon$ is being evaluated on some (k,q)-class $\mathcal{B}$. To this end, define

$(k{+}1,q)$- resp. $(k{-}1,q)$-classes $\mathcal{B}_{+1}, \mathcal{B}_{-1}$ by

$$\begin{aligned}
\mathcal{B}_{+1} &:= \{(\psi_0 + t\psi_k, \psi_1, \dots, \psi_k, \psi_k) \mid (\psi_0, \dots, \psi_k) \in \mathcal{B},\ 0 \le t \le 1\},\\
\mathcal{B}_{-1} &:= \{(\psi_0 + t\psi_k, \psi_1, \dots, \psi_{k-1}) \mid (\psi_0, \dots, \psi_k) \in \mathcal{B},\ 0 \le t \le 1\}
\end{aligned}$$

to provide the appropriate arguments for $\partial^\alpha \mathrm{d}_1^{k+1} R_\varepsilon$ resp. $\partial^\alpha \mathrm{d}_1^{k-1} R_\varepsilon$. On the basis of this choice of $\mathcal{B}_{+1}, \mathcal{B}_{-1}$ the proof of Theorem 18 of [28] can be upgraded by introducing ε as additional parameter throughout as to establish (1°_{A}) $\Rightarrow$ (2°_{A}) of the theorem.

Note that in the proof of (0°_{A}) $\Rightarrow$ (1°_{A}) being obtained from the proof of Theorem 13.1 by introducing the parameter ε, Theorem **(T4)** which, in turn, is based on Theorem 17.4 has to be invoked to guarantee the moderateness of $\partial_i R$.

The last statement of the theorem follows from Theorem 17.5 since the corresponding statement thereof contains, among others, α and k as free variables. $\square$

A glance at the preceding proof shows virtually all the substantial results of this article to be involved. Note that a $(0,q)$-class consisting of a single element ϕ is nothing else than (a singleton containing) a test object of type $[\varepsilon, \mathrm{A}]_q$. The equivalence of $R \in \mathcal{N}[\mathrm{A}_\mathrm{g}^\infty]$ and condition (0°_{A}) without the uniformity clause (provided $R \in \mathcal{E}_M[\mathrm{A}_\mathrm{g}^\infty]$) now shows that for a function $R \in \mathcal{E}(\Omega)$ which is moderate with respect to type $[\varepsilon x, \mathrm{A}_\mathrm{g}^\infty]$, it amounts to the same to be negligible with respect to either type $[\varepsilon x, \mathrm{A}_\mathrm{g}^\infty]$ or $[\varepsilon, \mathrm{A}]$. We will make use of this fact below. For the following theorem, recall that $\mathcal{G}_0^e$ denotes the smooth part of $\mathcal{G}^e$ (cf. the discussion following Definition 16.3).

17.10 Theorem. *Of the canonical maps* $\mathcal{G}^d(\Omega) \to \mathcal{G}^2(\Omega) \to \mathcal{G}^1(\Omega) \to \mathcal{G}_0^e(\Omega)$ *the first and the second one are injective whereas the third one is not. The four corresponding spaces of representatives (i.e., of moderate functions) are pairwise different.*

Proof. The injectivity of the map $\mathcal{G}^d(\Omega) \to \mathcal{G}^2(\Omega)$ is equivalent to $\mathcal{N}[\varepsilon x, \mathrm{V}] \cap \mathcal{E}_M[\varepsilon x, 0] = \mathcal{N}[\varepsilon x, \mathrm{A}_{\mathrm{g}}^\infty] \cap \mathcal{E}_M[\varepsilon x, 0]$. This, however, is accomplished by the extension $(3^\circ) \Leftrightarrow (5^\infty)$ of Theorem 7.9 derived from the first diagram in section 16. The injectivity of $\mathcal{G}^2(\Omega) \to \mathcal{G}^1(\Omega)$, on the other hand, is equivalent to $\mathcal{E}_M[\varepsilon x, \mathrm{A}_{\mathrm{g}}^\infty] \cap \mathcal{N}[\varepsilon, \mathrm{A}] = \mathcal{N}[\varepsilon x, \mathrm{A}_{\mathrm{g}}^\infty]$ which has been deduced previously from Theorem 17.9. Finally, to establish the non-injectivity of $\mathcal{G}^1(\Omega) \to \mathcal{G}_0^e(\Omega)$, we use the fact that the function P introduced in section 15 can be shown not to be negligible with respect to type $[\varepsilon, \mathrm{A}]$, by techniques similar to those employed in section 15. The difference of the respective spaces of moderate functions should be clear from the following examples. $\square$

17.11 Examples. Let $\Omega := \mathbb{R}$.

(i) Both $R_2(\varphi, x) := \exp(\langle\varphi|\varphi\rangle^2 \langle\xi, \varphi(\xi)\rangle)$ and $R_3(\varphi, x) := \exp(\varphi(0)^2 \langle\xi, \varphi\rangle)$ are moderate of type $[\varepsilon x, \mathrm{A}_{\mathrm{g}}^\infty]$, yet not of type $[\varepsilon x, 0]$.

(ii) The counterexample $R_0(\varphi, x) := \exp(i \exp(\langle\varphi|\varphi\rangle))$ (see [28]) which has already been mentioned in connection with the failure of algebras of type $[\varepsilon]$ to be diffeomorphism invariant (see 16.4(i)) has the property of being moderate of type $[\varepsilon, \mathrm{A}]$ yet not of type $[\varepsilon x, \mathrm{A}_{\mathrm{g}}^\infty]$. The same holds true for R_5 to be defined below.

(iii) $R_4(\varphi, x) := \langle\xi, \varphi(\xi)\rangle \cdot \exp(\varphi(0))$ is moderate of type $[\mathrm{c}, \mathrm{V}]$, yet not of type $[\varepsilon, \mathrm{A}]$. This also holds true for R_1 introduced as Example 16.4 (ii).

(iv) $R_5(\varphi, x) := \exp(-\langle\varphi|\varphi\rangle) \cdot \exp(i \exp(2\langle\varphi|\varphi\rangle))$ is of particular interest: It is not moderate with respect to any of the types $[\varepsilon x]$, however, it is even negligible with respect to all types $[\varepsilon]$ and $[\mathrm{c}]$.

Again the proofs of the preceding claims are left to the reader.

It is a remarkable fact that answering the apparently harmless question of injectivity of the canonical maps in the last analysis involves quite a number of hard theorems: the extension $(3^\circ) \Leftrightarrow (5^\infty)$ of Theorem 7.9 derived in section 16; Theorem 17.9 which, in turn, is based on part $(1^\circ) \Leftrightarrow (2^\circ)$ Theorem 18 of [28] and on Theorems 13.1, 17.4

and 17.5; finally, also the counterexample P of section 15 is among the ingredients of the argument. It remains to prove Theorems 17.4 and 17.5.

Proof of Theorem 17.4. To show sufficiency of the condition, suppose that the differentials of R_ε (where $R \in \mathcal{E}(\Omega)$) satisfy the property specified in the theorem. Consider a test object $\phi(\varepsilon, x)$ of type $[\mathrm{A}_\mathrm{g}^\infty]_N$ and set $\Phi(\varepsilon, x) := (\phi(\varepsilon, x), x)$. Expanding $\partial^\alpha(R_\varepsilon \circ \Phi)$ according to the chain rule shows that R is moderate of type $[\mathrm{A}_\mathrm{g}^\infty]$: It suffices to observe that the family of all finite sequences

$$(\phi(\varepsilon, y), \partial_y^{\beta_1}\phi(\varepsilon, y), \ldots, \partial_y^{\beta_l}\phi(\varepsilon, y))$$

forms an (l, N)-class if ε is considered as variable and y as a parameter taking values in some compact subset of Ω.

Conversely, for a function $R \in \mathcal{E}(\Omega)$ which is moderate with respect to type $[\mathrm{A}_\mathrm{g}^\infty]$, we will show that the assumption of R to violate the condition in the theorem leads to a contradiction. Thus suppose that there exist $K \subset\subset \Omega$, $\alpha \in \mathbb{N}_0^s$, $k \in \mathbb{N}_0^s$ such that for all $N \in \mathbb{N}$ there exists a (k, N)-class $\mathcal{B}$ such that

$$\sup_{K,\mathcal{B}} |\partial^\alpha \mathrm{d}_1^k R_\varepsilon(\psi_0(\varepsilon), x)(\psi_1(\varepsilon), \ldots, \psi_k(\varepsilon))| \tag{10}$$

is *not* of order ε^{-N}. By moderateness of R, there exists $N \in \mathbb{N}$ such that

$$\sup_K |\partial^{\alpha'}(R_\varepsilon(\phi(\varepsilon, x), x))| = O(\varepsilon^{-N}) \tag{11}$$

for all test objects ϕ of type $[\mathrm{A}_\mathrm{g}^\infty]_N$, where $\alpha' := \alpha + pe_s$, $p := \sum_{i=1}^{k}(|\alpha| + k^2 + i)$. Due to our hypothesis, there exists a (k, N)-class $\mathcal{B}$ such that (10) is not of order ε^{-N}. Having fixed $K, \alpha, k, N, \mathcal{B}$, we inductively define sequences $x^{(j)} \in K$, $(\psi_0^{(j)}, \ldots, \psi_k^{(j)}) \in \mathcal{B}$, $0 < \varepsilon_j < \frac{1}{j}$ (with $\varepsilon_{j+1} < \varepsilon_j$) $(j = 1, 2, \ldots)$ such that the following inequalities hold for $j = 1, 2, \ldots$:

$$|\partial^\alpha \mathrm{d}_1^k R_{\varepsilon_j}(\psi_0^{(j)}(\varepsilon_j), x^{(j)})(\psi_1^{(j)}(\varepsilon_j), \ldots, \psi_k^{(j)}(\varepsilon_j))| \geq j \cdot \varepsilon_j^{-N} \tag{12}$$

(the technical details are similar to those in the proof of part (C)⇒(A) of Theorem 10.5). Let $(\lambda_j)_{j\in\mathbb{N}}$ be a partition of unity on I as in Lemma 10.1; for $(t_1, \ldots, t_k) \in \{0, 1, \ldots, k\}^k$, define

$$\phi_{t_1,\ldots,t_k}(\varepsilon, x) := \sum_{j=1}^{\infty} \lambda_j(\varepsilon) \cdot \left[\psi_0^{(j)}(\varepsilon) + \sum_{i=1}^{k} t_i \frac{(x_s - x_s^{(j)})^{|\alpha|+k^2+i}}{(|\alpha| + k^2 + i)!} \cdot \psi_i^{(j)}(\varepsilon)\right].$$

Since $\sum_{i=1}^{k} t_i \frac{(x_s - x_s^{(j)})^{|\alpha|+k^2+i}}{(|\alpha|+k^2+i)!}$ is a polynomial in x and all $(\psi_0^{(j)}, \ldots, \psi_k^{(j)})$ are members of one particular (k, N)-class $\mathcal{B}$, $\phi_{t_1,\ldots,t_k}$ is a member of $\mathcal{C}_b^\infty(I, \mathcal{A}_0(\mathbb{R}^s))$ and, in addition,

is of type $[\mathrm{A}_{\mathrm{g}}^{\infty}]_N$. From (11) we conclude that

$$\sup_K |\partial^{\alpha'}(R_\varepsilon(\phi_{t_1,\dots,t_k}(\varepsilon,x),x))| = O(\varepsilon^{-N}). \tag{13}$$

Now we follow the combinatorial reasoning of the proof of Theorem 17 of [28] to derive the desired contradiction: Choosing numbers $c_0,\dots,c_k$ satisfying the set of equations $\sum\limits_{i=0}^{k} c_i \cdot i^m = \delta_{1m}$ $(m = 0,1,\dots,k)$, let us form

$$\sum_{t_1=0}^{k} \dots \sum_{t_k=0}^{k} c_{t_1} \dots c_{t_k} \partial^{\alpha'}(R_\varepsilon(\phi_{t_1,\dots,t_k}(\varepsilon,x),x)). \tag{14}$$

By (13), this expression is of order ε^{-N}, uniformly for $x \in K$. On the other hand, evaluating (14) at $\varepsilon := \varepsilon_j$, $x := x^{(j)}$ according to the chain rule results in a positive integer multiple of $\partial^\alpha \mathrm{d}_1^k R_{\varepsilon_j}(\psi_0^{(j)}(\varepsilon_j), x^{(j)})(\psi_1^{(j)}(\varepsilon_j),\dots,\psi_k^{(j)}(\varepsilon_j))$, due to the delicate combinatorial argument of the proof of Theorem 17 of [28]. (14) being of order ε^{-N}, we conclude that

$$|\partial^\alpha \mathrm{d}_1^k R_{\varepsilon_j}(\psi_0^{(j)}(\varepsilon_j), x^{(j)})(\psi_1^{(j)}(\varepsilon_j),\dots,\psi_k^{(j)}(\varepsilon_j))| \le C\varepsilon_j^{-N} \qquad (j \ge j_0)$$

for some positive constant $C > 0$ and some $j_0 \in \mathbb{N}$. This, however, contradicts our choice of $x^{(j)}$, $\psi_i^{(j)}$. So the condition in the theorem, in fact, is necessary for R being moderate. (In a trivial way, the preceding reasoning also applies in the case $k = 0$ if all sums $\sum\limits_{i=1}^{k}$ are set equal to 0.)

To prove the last statement of the theorem, let K, α, k, N be given. Suppose, again by way of contradiction, the condition on the differentials of R_ε given in the theorem to be satisfied for (k,N)-classes consisting of a single element, yet to be violated for arbitrary (k,N)-classes, either with resepect to the particular K, α, k, N at hand. Similarly to the reasoning of the main part of the proof, deduce from these hypotheses the existence of a (k,N)-class $\mathcal{B}$ and of sequences $0 < \varepsilon_{j+1} < \varepsilon_j < \frac{1}{j}$, $x^{(j)} \in K$, $(\psi_0^{(j)},\dots,\psi_k^{(j)}) \in \mathcal{B}$ $(j = 1,2,\dots)$ satisfying the inequalities (12) for all $j \in \mathbb{N}$. Now define

$$\psi_i(\varepsilon) := \sum_{j=1}^{\infty} \lambda_j(\varepsilon)\psi_i^{(j)}(\varepsilon) \qquad (i = 0,1,\dots,k)$$

where $(\lambda_j)_{j\in\mathbb{N}}$ is a partition of unity on I as in Lemma 10.1. Due to the properties of the $\psi_i^{(j)}$, $\{(\psi_0,\dots,\psi_k)\}$ is a (k,N)-class. By assumption,

$$\sup_K |\partial^\alpha \mathrm{d}_1^k R_\varepsilon(\psi_0(\varepsilon),x)(\psi_1(\varepsilon),\dots,\psi_k(\varepsilon))| = O(\varepsilon^{-N}).$$

Taking into account that $\psi_i(\varepsilon_j) = \psi_i^{(j)}(\varepsilon_j)$, this contradicts our choice of $x^{(j)}$, $\psi_i^{(j)}$, thereby completing the proof. $\square$

Proof of Theorem 17.5. Just copy the proof of Theorem 17.4, add "for all $n \in \mathbb{N}$" at the appropriate places and change "ε^{-N}" to "ε^{n}". At the remaining occurrences of N, replace it by q. $\square$

18 Concluding remarks

As has been pointed out already in Part I, Theorem 7.14 has a place at the very core of diffeomorphism invariance of a Colombeau algebra. The problem with algebras of any type $[\mathrm{c}, Y]$, of course, is that classes consisting of test objects as simple as $\varphi \in \mathcal{A}_0(\mathbb{R}^s)$ are not invariant under the action of a diffeomorphism since the latter introduces dependence on ε and x. Types $[\varepsilon x, Y]$ of course are an efficient remedy against that problem as they incorporate a very general (ε, x)-dependence into test objects. Yet there is an intermediate way: Starting with the class of "constant" test objects $\tilde{\varphi} \in \mathcal{A}_0(\mathbb{R}^s)$ resp. $\in \mathcal{A}_q(\mathbb{R}^s)$, we consider the minimal class containing these which is invariant with respect to diffeomorphisms. Due to the functorial property of $\bar{\mu}_\varepsilon$, this class precisely consists of all images of constant test objects (in the sense just described) under the mappings $\tilde{\varphi} \mapsto ((\varepsilon, x) \mapsto \mathrm{pr}_1\, \bar{\mu}_\varepsilon(\tilde{\varphi}, \mu^{-1}x))$ where μ ranges over all diffeomorphisms onto the open set under consideration. As the following example shows, the class of test objects obtained in this way (starting with all $\tilde{\varphi} \in \mathcal{A}_0(\mathbb{R}^s)$) is in fact different in general from $\mathcal{C}_b^\infty(I \times \Omega, \mathcal{A}_0(\mathbb{R}^s))$.

18.1 Example. Let $\Omega := \mathbb{R}$; choose $\psi \in \mathcal{A}_0(\mathbb{R})$ satisfying $\psi(0) \neq 0$, $\operatorname{supp}\psi \subseteq [-1, +1]$. Setting $\phi(\varepsilon, x)(\xi) := \psi(\xi + \sin x)$ in fact defines an element of $\mathcal{C}_b^\infty(I \times \mathbb{R}, \mathcal{A}_0(\mathbb{R}))$. Now assume that there exists $\tilde{\varphi} \in \mathcal{A}_0(\mathbb{R})$ and a diffeomorphism $\mu : \tilde{\Omega} \to \mathbb{R}$ (where $\tilde{\Omega} \subseteq \mathbb{R}$ is open) such that $(\phi(\varepsilon, x), x) = \bar{\mu}_\varepsilon(\tilde{\varphi}, \mu^{-1}x)$ for all $x \in \mathbb{R}$. Setting $\xi := 0$, $x_1 := 0$, $x_2 := \frac{\pi}{2}$, respectively, we obtain $\psi(0) = \tilde{\varphi}(0) \cdot |(\mu^{-1})'(\mu(0))|$ resp. $\psi(1) = \tilde{\varphi}(0) \cdot |(\mu^{-1})'(\mu(\frac{\pi}{2}))|$. The first of these relations entails $\tilde{\varphi}(0) \neq 0$ while the second one implies $\tilde{\varphi}(0) = 0$, so we arrive at a contradiction.

In some situations, it may not even be desirable to require invariance of a Colombeau algebra with respect to all diffeomorphisms; for example, invariance only with respect to members of the Poincaré group might be of interest in applications in special relativity. This approach also could be combined with restricting the class of test objects to images of constant test objects under the particular group of transformations at hand. This opens the way to new classes of Colombeau algebras possessing weaker invariance properties than $\mathcal{G}^d(\Omega)$ does. However, these new objects still can be constructed on the basis of the scheme outlined in section 3 which, in our view,

constitutes an appropriate general framework for the treatment of (full) Colombeau algebras.

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References

[1] H. Balasin, *Colombeau's generalized functions on arbitrary manifolds,* gr-qc Preprint[1] **9610017** (1996).

[2] H. Balasin, *Distributional aspects of general relativity: The expample of the energy-momentum tensor of the extended Kerr-geometry,* in M. Grosser, G. Hörmann, M. Kunzinger, M. Oberguggenberger (Eds.), Nonlinear Theory of Generalized Functions, Chapman & Hall/CRC Res. Notes Math. **401**, Chapman & Hall/CRC, Boca Raton, FL, 1999, pp. 275–290.

[3] H. A. Biagioni, *A Nonlinear Theory of Generalized Functions,* (2nd ed.) Lecture Notes in Math. **1421**, Springer, New York, 1990.

[4] H. A. Biagioni, M. Oberguggenberger, *Generalized solutions to the Korteweg-de Vries and the regularized long-wave equations,* SIAM J. Math. Anal. **23** (1992), 923–940.

[5] H. A. Biagioni, M. Oberguggenberger, *Generalized solutions to Burgers' equation,* J. Differential Equations **97** (1992), 263–287.

[6] F. Berger, J. F. Colombeau, *Numerical solutions of one-pressure models in multifluid flows,* SIAM J. Numer. Anal. **32** (1995), 1139–1154.

[7] F. Berger, J. F. Colombeau, M. Moussaoui, *Solutions mesures de Dirac de systemes de lois de conservation et applications numeriques,* C. R. Acad. Sci. Paris Sér. I Math. **316** (1993), 989–994.

[8] J. F. Colombeau, *Differential Calculus and Holomorphy, Real and Complex Analysis in Locally Convex Spaces,* North Holland, Amsterdam, 1982.

[1]available electronically under `http://xxx.lanl.gov/archive/gr-qc`

[9] J. F. Colombeau, *New Generalized Functions and Multiplication of Distributions,* North Holland, Amsterdam, 1984.

[10] J. F. Colombeau, *Elementary Introduction to New Generalized Functions,* North Holland, Amsterdam, 1985.

[11] J. F. Colombeau, *Multiplication of distributions,* Bull. Amer. Math. Soc. (N.S.) **23** (1990), 251–268.

[12] J. F. Colombeau, *Multiplication of Distributions. A Tool in Mathematics, Numerical Engineering and Theoretical Physics,* Lecture Notes in Math. **1532**, Springer, New York, 1992.

[13] J. F. Colombeau, A. Meril, *Generalized functions and multiplication of distributions on $\mathcal{C}^\infty$ manifolds,* J. Math. Anal. Appl. **186** (1994), 357–364.

[14] J. F. Colombeau, A. Heibig, M. Oberguggenberger, *Le probleme de Cauchy dans un espace de fonctions generalisees I,* C. R. Acad. Sci. Paris Sér. I Math. **317** (1993), 851–855.

[15] J. F. Colombeau, A. Heibig, M. Oberguggenberger, *Le probleme de Cauchy dans un espace de fonctions generalisees II,* C. R. Acad. Sci. Paris Sér. I Math. **319** (1994), 1179–1183.

[16] J. F. Colombeau, M. Oberguggenberger, *On a hyperbolic system with a compatible quadratic term: Generalized solutions, delta waves, and multiplication of distributions,* Comm. Partial Differential Equations **15** (1990), 905–938.

[17] J. W. de Roever, M. Damsma, *Colombeau algebras on a $\mathcal{C}^\infty$-manifold,* Indag. Math. (N.S.) **2** (1991), 341–358.

[18] N. Dapić, S. Pilipović, *Microlocal analysis of Colombeau's generalized functions on a manifold,* Indag. Math. (N.S.) **7** (1996), 293–309.

[19] N. Dapić, S. Pilipović, D. Scarpalézos, *Microlocal analysis of Colombeau's generalized functions—Propagation of singularities,* J. Anal. Math. **75** (1998), 51–66.

[20] J. Dieudonné, *Éléments d'Analyse,* Vol **3**, Gauthier-Villars, Paris, 1974.

[21] E. Farkas, M. Grosser, M. Kunzinger, R. Steinbauer, *On the foundations of nonlinear generalized functions I,* Preprint, Wien, 1999.

[22] E. Farkas, *Approximation properties of convenient vector spaces,* Preprint, Wien, 1996. [1]

[23] A. Frölicher, A. Kriegl, *Linear Spaces and Differentiation Theory,* Wiley, Chichester, 1988.

[24] H. Grosse, M. Oberguggenberger, I. T. Todorov, *Generalized functions for quantum fields obeying quadratic exchange relations,* ESI Preprint[2] **653**, (1999).

[25] [Void; in the first part of the series this item refers to the present article.]

[26] M. Grosser, M. Kunzinger, R. Steinbauer, J. Vickers, *A global theory of nonlinear generalized functions,* Preprint, Wien, 1999.

[27] L. Hörmander, *The Analysis of Linear Partial Differential Operators I,* Grundlehren Math. Wiss. **256**, Berlin 1990.

[28] J. Jelínek, *An intrinsic definition of the Colombeau generalized functions,* Comment. Math. Univ. Carolin. **40** (1999), 71–95.

[29] B. L. Keyfitz, H. C. Kranzer, *Spaces of weighted measures for conservation laws with singular shock solutions,* J. Differential Equations **118** (1995), 420–451.

[30] A. Kriegl, P. W. Michor, *The Convenient Setting of Global Analysis,* Math. Surveys Monogr. **53**, Amer. Math. Soc., Providence, RI, 1997.

[31] M. Kunzinger, *Lie Transformation Groups in Colombeau Algebras,* doctoral thesis, University of Vienna, 1996.

[32] M. Kunzinger, R. Steinbauer, *A rigorous solution concept for geodesic and geodesic deviation equations in impulsive gravitational waves,* J. Math. Phys. **40** (1999), 1479–1489.

[33] E. Landau, *Einige Ungleichungen für zweimal differentiierbare Funktionen,* Proc. London Math. Soc. Ser. 2, **13** (1913–1914), 43–49.

[34] M. Nedelkov, S. Pilipović, D. Scarpalézos, *The Linear Theory of Colombeau Generalized Functions,* Pitman Res. Notes Math. Ser. **385**, Longman, Harlow, 1998.

[35] M. Oberguggenberger, *Multiplication of Distributions and Applications to Partial Differential Equations,* Pitman Res. Notes Math. Ser. **259**, Longman, Harlow, 1992.

[1] available electronically under `http://diana.mat.univie.ac.at/~diana/dianapub.html`

[2] available electronically under `http://www.esi.ac.at/ESI-Preprints.html`

[36] M. Oberguggenberger, F. Russo, *Nonlinear SPDEs, Colombeau solutions and pathwise limits,* in L. Decreusefond, J. Gjerde, B. Øksendal, A. S. Üstünel (Eds.), Stochastic Analysis and Related Topics VI., Birkhäuser, Boston, 1998, pp. 319–332.

[37] E. E. Rosinger, *Distributions and Nonlinear Partial Differential Equations,* Lecture Notes Math. **684**, Springer, New York, 1978.

[38] E. E. Rosinger, *Non-Linear Partial Differential Equations. An Algebraic View of Generalized Solutions,* North Holland, Amsterdam, 1990.

[39] H. H. Schaefer, *Topological Vector Spaces* (5th ed.), Grad. Texts in Math., Springer, 1986.

[40] L. Schwartz, *Sur l'impossibilite de la multiplication des distributions,* C. R. Acad. Sci. Paris Sér. I Math. **239** (1954), 847–848.

[41] R. Steinbauer *Distributional Methods in General Relativity,* doctoral thesis, University of Vienna, 1999.

[42] J. A. Vickers, J. P. Wilson, *Invariance of the distributional curvature of the cone under smooth diffeomorphisms,* Classical Quantum Gravity **16** (1999), 579–588.

[43] J. A. Vickers, J. P. Wilson, *A nonlinear theory of tensor distributions,* ESI Preprint[1] **566** (1998).

[44] J. A. Vickers, *Nonlinear generalised functions in general relativity,* in M. Grosser, G. Hörmann, M. Kunzinger, M. Oberguggenberger (Eds.), Nonlinear Theory of Generalized Functions, Chapman & Hall/CRC Res. Notes Math. **401**, Chapman & Hall/CRC, Boca Raton, FL, 1999, pp. 275–290.

[45] J. P. Wilson, *Distributional curvature of time dependent cosmic strings,* Classical Quantum Gravity **14** (1997), 2485–2498.

[46] S.Yamamuro, *Differential Calculus in Topological Linear Spaces,* Lecture Notes in Math. **374**, Springer, New York, 1974.

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[1] available electronically under `http://www.esi.ac.at/ESI-Preprints.html`